ON INTEGRALS RELATED TO AND EXTENSIONS OF THE LEBESGUE INTEGRALS.

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In the theory of definite integration, the central position is occupied by the definitions of integration due to Riemann and to Lebesgue. There have however been given a number of other definitions, some of which are related to and in part equivalent to the Lebesgue integral, and some of which are extensions of it; viz., in the direction of giving an integral for a function for which the Lebesgue definition is not applicable either on account of the fact that the set of points at which it is defined is not measurable, or on account of the fact that the function is not absolutely integrable; and in the direction of integration with respect to functions of bounded variation. We purpose in this paper to discuss briefly some of the definitions of integration which have been proposed and consider their relations to the Lebesgue integral. We shall divide the work into four sections.

In the first section we discuss the types of definition of integration which are extensions of the Darboux upper and lower integral method of defining a Riemann integral. Essentially the process involves three steps, (*a*) the selection of a simple class of functions and the definition of integration for functions of this class, (*b*) the extension of this class of functions by the addition of functions which are the limits of sequences converging either uniformly or monotonically, the integral of the limit being defined to be the limit of the integrals, (*c*) the further extension of this class by the addition of all functions $f(x)$ for which there exist functions φ and ψ belonging to the classes defined in (*a*) or (*b*), with equal definite integrals, and such that $\varphi \leqq f \leqq \psi$. We find ideas of this kind in the Lebesgue integration, the last of these steps suggesting, in particular, the distinguishing feature between Borel and Lebesgue measurable sets. Step (*b*), when the convergence is uniform, is present in the Lebesgue definition, while, when we take rather the monotonic sequence, we have

the ideas underlying the Young definition which we consider here. This definition is, as we shall show, equivalent to that of Lebesgue in the sense that every function integrable according to either definition is integrable according to the other and the values of the integrals are the same. Very much of the same order is the definition due to Pierpont, which differs from the definition of Young only in that it gives an integral for functions defined on non-measurable sets—assuming that such sets exist. We point out that the Pierpont integral is an extension only in a limited way, in that the functions which are integrable according to this definition are non-measurable in the sense that they are defined on non-measurable sets; in particular we show that if f is Pierpont integrable on a non-measurable set E then there exists a measurable set M containing E and a function φ defined on M which is equal to f on E, such that the Lebesgue integral of φ on M is the same as the Pierpont integral of f on E. Perhaps by stretching the meaning of the word equivalence a little, we might say that the Pierpont and Lebesgue integrals are equivalent.

The second section is devoted to a consideration of the definitions of the integrals of functions which are not integrable according to Lebesgue's definition, because the latter are always absolutely integrable. We consider two types of definitions. The first type depends upon the ability to find a set of intervals whose total length may be made to approach zero, and such that it is possible to define an integral for the function when defined only on the complementary set of points. If these latter integrals approach a limit as the total length of the complementary intervals approaches zero, the limit is the integral over the interval. The integral is thus defined as a result of a single limiting process, and all the singularities are treated simultaneously. The definitions of this type are the Jordan-Harnack-Moore definitions, in which the enclosing intervals are finite in number, and the Borel definitions, in which the intervals may be denumerably infinite in number, i. e., the first are based on content and the second on measure. The other type of definition depends upon the arrangement of the singularities of the function to be integrated in some order, the arrangement and the singularities being dependent upon the integration process used, the ultimate value of the integral being obtained by at most a denumerable infinity of steps and limiting processes. In this cate-

gory belong the Dirichlet extension of the Cauchy integral for unbounded functions, and the extension of this to the case when the points in the vicinity of which $f(x)$ is not bounded form a reducible set. It is this idea also which Denjoy has applied successfully in order to obtain a definition of integral of unbounded functions which includes as special cases practically all those which have been previously suggested, an integral which proves of importance in the treatment of the relation between a continuous function and its derivative.

The third section is devoted to the Stieltjes integral, which has recently come into the foreground on account of the rôle which it plays in the theory of linear functional operations on continuous functions. We point out that a Stieltjes integral is expressible in terms of a Lebesgue integral of another function, and conversely, but that in spite of this, the Stieltjes integral seems to be applicable where the Lebesgue is not. We give an extension of the Stieltjes integral modelled on the Lebesgue extension of the Riemann integral, as well as the Fréchet generalization of the Lebesgue and Stieltjes integral, so as to apply to a class of general elements. This latter integral depends for its definition on the existence of an absolutely additive function whose range is a set of subclasses of the fundamental class.

The last section gives the definition of the Hellinger integral and also the generalizations of this due to Radon and E. H. Moore.

In the course of the paper we have occasion to point out certain equivalences. They are of two types. The first might be called a complete equivalence, i. e., one in which two definitions of integration yield the same class of integrable functions, and give the same value when applied to a given function. Of this character are the equivalence between the Young and Lebesgue, and one of the Borel definitions and that of Lebesgue;—and we might call attention in passing to what seems to be an indication of the felicity of the Lebesgue definition, that it can be approached from so many distinct and interesting points of view. The other type of equivalence is that which arises in connection with the Stieltjes and Hellinger integrals, each of which is by a transformation reducible to a Lebesgue integral, and conversely; but in either case the function to be integrated is no longer the same. We are inclined to consider this type of equivalence in the light of a

pseudo-equivalence, in so far as it is rather a reduction from one form to the other for purposes of evaluation.

The extension of the Lebesgue integral idea in the future will undoubtedly be in the direction of the formulation of a satisfactory integral operation for the case in which the field of integration is a space of a denumerable infinity of dimensions, and the function space. Thus far, there have been given two suggestions for such operations. One is due to Fréchet, who forms, for a general class, the natural extension of the Lebesgue and Stieltjes definition of integration. It depends upon the existence, in a general class of elements, of an absolutely additive function v whose range is a class of subclasses of the fundamental class, i. e., a function such that

$$v(\Sigma_n E_n) = \Sigma_n v(E_n),$$

the E_n being mutually distinct and finite or denumerably infinite in number. The examples of this which have been given for the general space are trivial in that they reduce either to an infinite sum or an integral extended over a field in a finite number of dimensions. There is still lacking a really effective and desirable absolutely additive function for the higher type of spaces. The other suggestion for generalization is due to E. H. Moore. It is essentially an operator of the bilinear or quadratic type, the Fréchet generalization being of the linear type. It depends upon the existence of a function $\epsilon(p, q)$ of two variables p, q, each of which ranges independently over the fundamental class of general elements, and satisfies the condition that for any finite set of elements $p_1, \cdots, p_n$, the quantities $\epsilon(p_i, p_j)$, $i, j = 1, \cdots, n$, form the coefficients of a positively definite Hermitian form in n variables. The operator is determined as a double least upper bound, and is theoretically given as soon as the proper type of ϵ is specified. Moore has given instances of this ϵ in various general spaces which are not trivial. While these two generalizations point the way in which one may go towards the generalization of the Lebesgue integral, they are not entirely satisfactory, the Fréchet generalization, as indicated above, because his instances are trivial, the Moore because it reduces to a Lebesgue integral only after a transformation. There is still room for considerable improvement and investigation in this field.

We restrict ourselves throughout this paper to functions

which are single-valued and take finite values at points belonging in most cases to a finite interval (a, b), or a set of points on such an interval. The functions are not, however, necessarily bounded on the interval (a, b). Many of the results are immediately extensible to fields of integration in n-dimensional space. Further, some of the theorems are still valid if $f(x)$ be definitely infinite or many-valued at a set of points of measure zero.

For the sake of convenience we have collected here the most important references which we have consulted in the compilation of this paper, and shall cite them only by number.

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I. THE YOUNG AND PIERPONT DEFINITIONS OF INTEGRATION.

1. *The Darboux Integrals.*—In so far as the Young and Pierpont definitions of integration which we consider in this chapter are extensions of the Darboux treatment of the Riemann integral by means of upper and lower integrals, we recall briefly the definition of these integrals for functions $f(x)$ bounded on an interval (a, b) and a few of their properties.

Definition.—Suppose a partition of the interval (a, b) has been effected by means of the points $a = x_0 < x_1 < \cdots < x_n = b$.

Let M_i be the least upper bound and m_i the greatest lower bound of the values of $f(x)$ on (x_{i-1}, x_i) and form the sums

$$S = \sum_{i=1}^{n} M_i(x_i - x_{i-1}) \qquad \text{and} \qquad s = \sum_{i=1}^{n} m_i(x_i - x_{i-1}).$$

Then the greatest lower bound of S and the least upper bound of s exist, if we consider all possible partitions of the interval (a, b) into a finite number of subintervals. These bounds are the upper and lower Darboux integrals, respectively, of $f(x)$ on (a, b) and are denoted by

$$\overline{\int_a^b} f(x)dx \qquad \text{and} \qquad \underline{\int_a^b} f(x)dx.$$

We note the following properties:

(1) If $m \leqq f(x) \leqq M$ for x on (a, b), then

$$m(b - a) \leqq s \leqq \underline{\int_a^b} f(x)dx \leqq \overline{\int_a^b} f(x)dx \leqq S \leqq M(b - a).$$

(2) The value of S is not increased, and that of s is not diminished by a repartition of a partition of (a, b).

(3) There exists a sequence of partitions of (a, b) each a repartition of the preceding and with norms approaching zero such that, if s_n and S_n are the corresponding sums s and S, we have

$$\lim_n s_n = \underline{\int_a^b} f(x)dx \qquad \text{and} \qquad \lim_n S_n = \overline{\int_a^b} f(x)dx.$$

Evidently $s_1 \leqq s_2 \leqq \cdots$ and $S_1 \geqq S_2 \geqq \cdots$. Moreover if $\varphi_n(x) = M_{n,i}$ and $\psi_n(x) = m_{n,i}$ in the ith interval $(x_{i-1} \leqq x < x_i)$ of the nth partition, then $\lim_n \varphi_n = \varphi$ and $\lim_n \psi_n = \psi$ exist and we have

$$\psi \leqq f \leqq \varphi; \qquad \text{and} \qquad \overline{\int} \varphi = \overline{\int} f \qquad \text{and} \qquad \underline{\int} \psi = \underline{\int} f.$$

(4) A necessary and sufficient condition that $f(x)$ be Riemann integrable is that $\underline{\int} = \overline{\int} = \int$.

From this, other necessary and sufficient conditions for Riemann integrability can be deduced. Perhaps the simplest is the following, due to Lebesgue ((13), page 29):

(5) A necessary and sufficient condition that $f(x)$ be Riemann integrable is that the points of discontinuity of $f(x)$ form a set of measure zero.

2. *The Young Definition of Integration.*—(Cf. Young (26), (27), (28); Nalli (16), pages 97–103.) The Riemann-Darboux definitions of integration depend upon a division of the interval of integration into subintervals; the Lebesgue definition on a division of the interval of variation of the function, which for the interval of integration results in a partition into measurable sets. The question naturally arises whether it is possible to find a definition of Lebesgue integral by replacing in the Darboux definition intervals by measurable sets.* Young ((26), page 243) therefore suggests the following

DEFINITION. *Divide the interval (a, b) into a finite or a denumerably infinite number of measurable sets E_i of measure d_i. Let M_i be the least upper bound and m_i the greatest lower bound of $f(x)$ on E_i and form the sums*

$$S = \Sigma_i M_i d_i \qquad \text{and} \qquad s = \Sigma_i m_i d_i.$$

Then the greatest lower bound of S and the least upper bound of s for all possible divisions of (a, b) into measurable sets are defined to be the Young (or (Y)) upper and lower integrals of $f(x)$ on (a, b). $f(x)$ is said to be Young (or (Y)) integrable if the upper and lower integrals are finite and equal, i. e.,

$$(Y)\int = (Y)\underline{\int} = (Y)\bar{\int}.$$

Obviously this definition is also applicable in case the interval (a, b) is replaced by any measurable set of points E.

This definition was originally suggested only for functions $f(x)$ bounded on (a, b). In that case the upper and lower integrals always exist, and we obtain the same value for each of them, whether the partition be into a finite or a denumerably infinite number of measurable sets. The same definition will apply also if $f(x)$† is not bounded, provided we assume that $f(x)$ is such that there exist partitions of (a, b) into a denumerable infinity of measurable sets on each of which $f(x)$

* De la Vallée Poussin (24), pp. 54–56, has pointed out that the Riemann integral is characterized by being a function of intervals, the Lebesgue integral being a function of measurable sets.

† While not expressed explicitly in Young's work, it seems that he considered this extension. Cf., for instance, (28), p. 35.

has a finite upper and lower bound. In determining the upper and lower integrals in this case, we restrict ourselves to partitions of this latter kind.

We are then able to show that, if the upper integral has not the value $-\infty$, the terms of the sum $\Sigma_i M_i d_i$, in which M_i is negative, form a convergent series, i. e., for every such partition, $\Sigma_i M_i d_i$ either diverges to $+\infty$ or converges absolutely. For, if the negative terms of this sum are divergent, then by suitable repartitions of (a, b) and rearrangement of the terms, we can make the sum approach $-\infty$. An analogous result holds for the lower integral, i. e., if it is not $+\infty$, then it either diverges to $-\infty$ or converges. It follows from this that a necessary and sufficient condition that the upper and lower integrals of $f(x)$ be finite is that there exist a partition of (a, b) such that if we form the sum $\Sigma_i M_i d_i$ for $|f(x)|$ this sum is convergent.*

In the following list of properties we exclude the case in which $(Y)\bar{\int} = -\infty$ or $(Y)\underline{\int} = +\infty$. Then the proofs are very much as in the case of the Darboux integrals, in so far as the series which enter will be absolutely divergent or convergent. We have

$$(1) \qquad (Y)\bar{\int} \geqq (Y)\underline{\int}.$$

If $f(x)$ is bounded, i. e., $m \leqq f \leqq M$, for every x on (a, b), then

$$m(b-a) \leqq (D)\underline{\int} \leqq (Y)\underline{\int} \leqq (Y)\bar{\int} \leqq (D)\bar{\int} \leqq M(b-a).$$

(2) The values of S are not increased, nor those of s diminished by a repartition of a partition of (a, b) into measurable sets.

(3) There exists a sequence of partitions of (a, b) into measurable sets, each a repartition of the preceding, such that if S_n and s_n are the corresponding values of S and s, then

$$\lim_n S_n = (Y)\bar{\int} \qquad \text{and} \qquad \lim_n s_n = (Y)\underline{\int}$$

and

$$S_1 \geqq S_2 \geqq \cdots \qquad \text{and} \qquad s_1 \leqq s_2 \leqq \cdots.$$

If we let $\varphi_n(x) = M_{n,i}$ and $\psi_n(x) = m_{n,i}$ on the ith set of

* Cf. Fréchet (6), p. 257.

the nth partition, then the functions $\varphi_n(x)$ will form a monotonic non-increasing sequence and the $\psi_n(x)$ a monotonic non-decreasing sequence of measurable functions such that

$$\lim_n \psi_n(x) = \psi(x) \leqq f(x) \leqq \varphi(x) = \lim_n \varphi_n(x).$$

Evidently the Lebesgue integrals of $\varphi_n(x)$ and $\psi_n(x)$ exist and are equal to the values of S_n and s_n respectively. Hence by the properties of Lebesgue (or (L)) integrals relative to monotonic sequences (cf. (13), page 98), we have

$$\lim_n S_n = \lim_n (L)\int \varphi_n = (L)\int \varphi = (Y)\overline{\int} f$$

and

$$\lim_n s_n = \lim_n (L)\int \psi_n = (L)\int \psi = (Y)\underline{\int} f.$$

3. *The Relation between the Young and Lebesgue Integrals.*— (Cf. Young (26), pages 243 ff.; Nalli (16), pages 97–106.) A direct consequence of (1) of § 2 is that if $f(x)$ is Riemann integrable, it is also Young integrable. More generally we have:

The Young and Lebesgue definitions of integration are equivalent, and the values obtained by the two definitions the same.

(a) Suppose $f(x)$ (L) integrable on (a, b). Let l_i $(i = -\infty$ to $+\infty)$ be points of division of the interval $(-\infty, +\infty)$. Then the sum

$$\sigma = \Sigma_i l_i \times \text{meas}\,(E(l_i \leqq f < l_{i+1}))$$

will be an s and the sum

$$\Sigma = \Sigma_i l_i \times \text{meas}\,(E(l_{i-1} < f \leqq l_i))$$

will be an S. Hence

$$\sigma \leqq (Y)\underline{\int} \leqq (Y)\overline{\int} \leqq \Sigma.$$

Since σ and Σ approach the $(L)\int$ as the maximum value of the difference $l_{i+1} - l_i$ approaches zero, it is apparent that

$$(L)\int = (Y)\underline{\int} = (Y)\overline{\int} = (Y)\int.$$

(b) On the other hand suppose that $f(x)$ is (Y) integrable on the interval (a, b). Then by property (3) of § 2

$$(L)\int\psi = (Y)\underline{\int} f = (Y)\int f = (Y)\bar{\int} f = (L)\int\varphi,$$

i. e., $(L)\int(\varphi - \psi) = 0$. Hence φ differs from ψ at most at a set of points of measure zero, and consequently, since $\psi \leqq f \leqq \varphi$ at every point, f differs from the measurable and (L) integrable functions φ and ψ at most at a set of measure zero, and hence is measurable and also (L) integrable.

Closely allied with the idea used in this last proof is the following definition of Lebesgue integration due to Riesz ((20), page 453).

The integral is defined first of all for functions which are constant on measurable sets of points. If $f(x)$ is equal to a_i on the set E_i of measure d_i, then

$$\int_a^b f(x)dx = \Sigma_i a_i d_i,$$

the series on the right being assumed to be absolutely convergent if infinite. If we extend this class of functions by the addition of all limits of uniformly convergent sequences of such functions, we get the class of Lebesgue integrable functions. The integral of the limiting function is of course, by definition, the limit of the integrals. This definition lends itself admirably to the proof of theorems of the nature of the Schwarz inequality for summable functions.

In reality the above proof suggests the extension of the class of functions constant on measurable subsets of (a, b) by the addition of the limits of monotonic sequences of these functions, together with the functions which lie between a limit φ of a non-decreasing sequence, and a limit ψ of a non-increasing sequence for which the integrals are equal in value. Young (cf. (27) and (28)) has developed the theory of Lebesgue integration on the basis of monotonic sequences as just sketched, excepting that the initial functions for which integration is defined are upper and lower semi-continuous.

4. *Excursus on Upper and Lower Measure.*—(Cf. Hausdorff (10), pages 408–411.) Before taking up the discussion of the Pierpont integral it may be advantageous to derive a few elementary properties relating to upper and lower measure. We recall that the upper measure of a set is the greatest lower bound of the total lengths of the sets of open intervals which are such that each interval contains at least one point

of the set; that the lower measure is the measure of an interval containing the set, diminished by the upper measure of the complementary set with respect to the interval.

If for two sets E_1 and E_2 we denote by $E_1 + E_2$ their least common superset, and by E_1E_2 their greatest common subset, i. e., their logical sum and product, respectively, we have the following fundamental formula on intervals I_1, I_2:

(1) meas $(I_1 + I_2)$ + meas I_1I_2 = meas I_1 + meas I_2,

which is immediately extensible to a finite or a denumerably infinite set of intervals. An immediate consequence of this formula is

(2) $\overline{\text{meas}}\,(E_1 + E_2) + \overline{\text{meas}}\,E_1E_2 \leqq \overline{\text{meas}}\,E_1 + \overline{\text{meas}}\,E_2$

and by taking complements with respect to an interval including both E_1 and E_2

(3) $\underline{\text{meas}}\,(E_1' + E_2') + \underline{\text{meas}}\,E_1'E_2' \geqq \underline{\text{meas}}\,E_1' + \underline{\text{meas}}\,E_2'$,

where $E_1' = CE_1$ and $E_2' = CE_2$; i. e., this formula holds for any E_1 and E_2. From these two inequalities we conclude at once that if E_1 and E_2 are measurable, then the sets $E_1 + E_2$ and E_1E_2 are also measurable and

(4) meas $(E_1 + E_2)$ + meas E_1E_2 = meas E_1 + meas E_2.

From (2) we conclude for every E_1 and E_2

(5) $\overline{\text{meas}}\,(E_1 + E_2) \leqq \overline{\text{meas}}\,E_1 + \overline{\text{meas}}\,E_2$;

and from (3), if E_1 and E_2 have no points in common,

(6) $\underline{\text{meas}}\,(E_1 + E_2) \geqq \underline{\text{meas}}\,E_1 + \underline{\text{meas}}\,E_2$.

Further, since for every E_1 and E_2

$$CE_1 \leqq C(E_1 + E_2) + E_2,$$

we obtain by applying (5)

$$\overline{\text{meas}}\,CE_1 \leqq \overline{\text{meas}}\,E_2 + \overline{\text{meas}}\,C(E_1 + E_2),$$

or

(7) $\underline{\text{meas}}\,E_1 + \overline{\text{meas}}\,E_2 \geqq \underline{\text{meas}}\,(E_1 + E_2)$.

Similarly if E_1 and E_2 are distinct, we can apply (6) to

$$CE_1 = C(E_1 + E_2) + E_2$$

and obtain

(8) $\overline{\text{meas}}\, E_1 + \underline{\text{meas}}\, E_2 \leqq \overline{\text{meas}}\, (E_1 + E_2)$.

If, then, E_1 and E_2 are distinct, we can write down the following series of inequalities:

$$(9)\quad \underline{\text{meas}}\, E_1 + \underline{\text{meas}}\, E_2 \leqq \underline{\text{meas}}\, (E_1 + E_2) \leqq \underline{\text{meas}}\, E_1 + \overline{\text{meas}}\, E_2 \leqq \overline{\text{meas}}\, (E_1 + E_2) \leqq \overline{\text{meas}}\, E_1 + \overline{\text{meas}}\, E_2.$$

5. *Relative Measurability; Separated Sets.*—(Cf. Hausdorff (10), pages 415–6; Pierpont (18), pages 366–370.) We define:

E_0 will be said to be *measurable relative to* E, if there exists a measurable set M such that $E_0 = ME$, i. e., E_0 is, so to speak, a section of the set E by a measurable set.

Evidently E is measurable relative to itself. Also, if E_0 is measurable relative to E, then $E - E_0$ will also be measurable relative to E. Finally by (4) of § 4, we have that *if E is measurable, then relative measurability reduces to ordinary measurability.*

We note the following propositions:

(1) If E is divided into a finite or a denumerable infinity of sets E_i measurable relative to E, then

$$\overline{\text{meas}}\, E = \Sigma_i\, \overline{\text{meas}}\, E_i \qquad \text{and} \qquad \underline{\text{meas}}\, E = \Sigma_i\, \underline{\text{meas}}\, E_i.$$

For the case of two sets E_1 and E_2 with $E_1 + E_2 = E$ and $E_1 = ME$, where M is a measurable set, we have by (2) of § 4

$$\overline{\text{meas}}\, M + \overline{\text{meas}}\, E \geqq \overline{\text{meas}}\, (M + E) + \overline{\text{meas}}\, ME = \overline{\text{meas}}\, (M + E_2) + \overline{\text{meas}}\, E_1,$$

and so by (8) of § 4

$$\geqq \underline{\text{meas}}\, M + \overline{\text{meas}}\, E_2 + \overline{\text{meas}}\, E_1.$$

Since M is measurable, we obtain by using (5) of § 4

$$\overline{\text{meas}}\, E = \overline{\text{meas}}\, E_1 + \overline{\text{meas}}\, E_2.$$

The property for lower measure can be obtained in an analogous way by using (3), (7), and (6) of § 4. The corresponding

properties for any finite number, or denumerable infinitude of sets, follow at once from a recurrent use of the result for the case of two sets, and the properties

$$\overline{\text{meas}}\ \Sigma E_i \leqq \Sigma_i\ \overline{\text{meas}}\ E_i;$$

and if $\lim_n E_n = 0$, then $\lim_n \underline{\text{meas}}\ E_n = 0$.

The following converse of this proposition holds:

(2) If E is divided into a finite or denumerable set of sets E_i for which we have $\overline{\text{meas}}\ E = \Sigma_i\ \overline{\text{meas}}\ E_i$, then the E_i are measurable relative to E and $\underline{\text{meas}}\ E = \Sigma_i\ \underline{\text{meas}}\ E_i$.

We indicate the proof for two sets only, the case for a denumerably infinite number of sets being easily deducible from this. Suppose $E = E_1 + E_2$, E_1 and E_2 being distinct, and suppose M_1 and M_2 are measurable sets such that E_1 is contained in M_1 and E_2 in M_2, and meas $M_1 = \overline{\text{meas}}\ E_1$ and meas $M_2 = \overline{\text{meas}}\ E_2$. Then by (4) of § 4

$$\overline{\text{meas}}\ E = \text{meas}\ M_1 + \text{meas}\ M_2 = \text{meas}\ (M_1 + M_2) + \text{meas}\ M_1M_2 \geqq \overline{\text{meas}}\ E + \text{meas}\ M_1M_2,$$

i. e., meas $M_1M_2 = 0$. Hence $\overline{\text{meas}}\ M_2E_1 = 0$, i. e., M_2E_1 is measurable. Consequently, since

$$(M_2 - M_2E_1)E = M_2E - M_2E_1 = E_2,$$

E_2 will be the set common to E and the measurable set $M_2 - M_2E_1$ and so measurable relative to E.

As a consequence of these two propositions we have

(3) If $E_1 \leqq E_2 \leqq \cdots$ are a sequence of sets, each containing the preceding, and each measurable relative to the succeeding set in the sequence, then every set as well as the difference of any two sets, will be measurable relative to $E = \lim_n E_n$.

(4) If M is a measurable set containing E such that meas $M = \overline{\text{meas}}\ E$, then for any division of E into sets measurable relative to E, there exists a division of M into measurable sets M_i such that

$$\text{meas}\ M_i = \overline{\text{meas}}\ E_i \qquad \text{and} \qquad E_i = M_iE.$$

Pierpont ((18), page 366) uses in his work the notion of *separated division*, defined as follows:

E is divided into the separated sets E_1 and E_2 if it is

possible to enclose E_1 and E_2 in measurable sets M_1 and M_2 in such a way that meas $M_1M_2 = 0$.*

Pierpont ((18), page 379) shows that if E_1 and E_2 are separated then

$$\overline{\text{meas}}\ E = \overline{\text{meas}}\ E_1 + \overline{\text{meas}}\ E_2.$$

By using proposition (2) and its proof, we then have:

A necessary and sufficient condition that E_1 and E_2 constitute a separated division of E is that E_1 and E_2 be measurable relative to E, or that

$$\overline{\text{meas}}\ E_1 + \overline{\text{meas}}\ E_2 = \overline{\text{meas}}\ (E_1 + E_2) = \overline{\text{meas}}\ E.$$

Consequently, we have also

$$\underline{\text{meas}}\ E_1 + \underline{\text{meas}}\ E_2 = \underline{\text{meas}}\ E.$$

The same result holds if the division is into a finite or denumerably infinite number of separated sets.

Evidently, then, *if E is measurable, any separated division of E is always into measurable sets.*†

We shall use the term relatively measurable in preference to separated, on account of the close relationship to measurable sets.

6. *The Pierpont Definition of Integration.*—(Cf. Pierpont (18), pages 371 ff.) We are now in a position to give the Pierpont extension of the Young definition of integration.

Suppose E is any set of points on a finite interval, and $f(x)$ defined on E. Divide E into a finite or denumerable set of sets E_i measurable relative to E, and let M_i be the least upper bound, and m_i the greatest lower bound of $f(x)$ on E_i. Form the sums $S = \Sigma_i M_i \overline{\text{meas}}\ E_i$ and $s = \Sigma_i m_i \overline{\text{meas}}\ E_i$. The greatest lower bound of S and the least upper bound of s for all possible divisions of E into sets measurable relative to E are the upper and lower integrals, respectively, of $f(x)$ on E. $f(x)$ is said to be (P) integrable on E if these two integrals are finite and equal, i. e.,

$$(P)\overline{\int_E} = (P)\underline{\int_E} = (P)\int_E.$$

* We shall assume that E_1 and E_2 are distinct. Pierpont allows E_1 and E_2 to overlap, but since meas $M_1M_2 = 0$ we have also meas $E_1E_2 = 0$, i. e., they overlap only to the extent of a set of measure zero. The results obtained are the same with either definition, the treatment being slightly simpler as here given.

† It seems that recognition of this fact would have obviated the recent Fréchet-Pierpont controversy in these columns, vol. 22, 295–302; vol. 23, 172–5. Cf. also W. A. Wilson, vol. 22, pp. 384–386.

Evidently when the set E is measurable this definition is identical with that of Young. Although stated only for bounded functions by Pierpont, it is immediately applicable to functions not bounded on E. As a matter of fact the remarks made in connection with the Young integral can be immediately transferred to the Pierpont integral if we replace measurable by relatively measurable, and measure by upper measure. In particular, if we say that $f(x)$ is *measurable relative to* E (Pierpont, separable, cf. (18), page 403), if the set E_l of points of E for which $f(x) > l$ is measurable relative to E, we can build up an integral along the lines of Lebesgue. If we call a function integrable in this way *summable relative to* E, we have the theorem:

(1) *Pierpont integrability on E is equivalent to summability relative to E, the values obtained for the two integrals being the same.**

We note further the property

(2) If E is divided into a set of sets E_i measurable relative to E, and if f is integrable on each E_i, then f is also integrable on E and $\int_E = \Sigma_i \int_{E_i}$.

On the other hand we have the following:

(3) If f is (P) integrable on E, and $E_0 < E$, then f is also (P) integrable on E_0.

This holds, if we restrict ourselves to (Y) integrals, only in case both E_0 and E are measurable.

On account of the fact that, for a division of the set E into sets measurable relative to E, we also have

$$\underline{\text{meas}}\ E = \Sigma_i\ \underline{\text{meas}}\ E_i,$$

we can obtain a definition of integration on sets E by replacing, in the Pierpont integral, upper measure by lower measure. If we call integrability after the manner of this last definition (P_0) integrability, then if $f(x)$ is (P) integrable on E it is also (P_0) integrable. The converse does not seem to hold, because a set of lower measure zero does not necessarily have zero measure.

While the Pierpont integral gives an integral for a function defined on a non-measurable set, it is an extension of the

* Cf. Lamond, *Transactions Amer. Math. Society*, vol. 16 (1915), p. 393, where we find a similar theorem for f bounded.

Lebesgue definition only in a limited sense. For if we enclose any non-measurable set E in a measurable set M of such a nature that meas $M = \overline{\text{meas}}\, E$, then there exists on M a function $f_0(x)$ measurable on M and equal to $f(x)$ on E, such that

$$(L)\int_M f_0(x) = (P)\int_E f(x);$$

i. e., a function $f(x)$ measurable relative to a set E is a section of a measurable function defined on a measurable set including E. By using (4) of § 5, we can construct the function $f_0(x)$ by taking the function φ of § 2, corresponding to f, but defined on M, and modifying it at the points of E of measure zero, in such a way that it agrees with f on E.

We gain insight into the Pierpont integral also from the geometrical point of view. If $f > 0$, and we consider the planar set [x on E, $0 \leqq y < f(x)$], then the (P) integral gives us the upper planar measure of this set, the (P_0) integral the lower planar measure. These will be equal for $f > 0$ only if the set E for which $f > 0$ is measurable, i. e., f is Lebesgue integrable.

The (P) integral has proved of value in connection with the reduction of a double integral to an iterated integral, in so far as a linear section of a planar measurable set is not necessarily linearly measurable.

It seems, however, that the question of giving a definition for the integration of essentially non-measurable functions, i. e., for instance functions defined on a measurable set which are not measurable, has not yet been satisfactorily solved.

7. *The Pierpont Definition of Integration for Unbounded Functions.*—(Cf. (8), pages 405 ff.) We note finally Pierpont's treatment of integrals of functions which are not bounded on a set E, very briefly. Let $E_{\alpha\beta}$ $(\alpha, \beta > 0)$ be the subset of E for which $-\beta \leqq f(x) \leqq \alpha$. Suppose that the $(P)\int_{E_{\alpha\beta}} f(x)dx$ exists for every $\alpha, \beta > 0$. Then $\int_E$ is defined to be the

$$\lim_{\alpha,\,\beta = \infty} \int_{E_{\alpha\beta}}$$

provided this limit exists and has a finite value.

This definition is equivalent to the one given above in § 6. For, if we assume that $(P)\int_E f$ exists, then f will be relatively measurable on E, and hence $(P)\int_{E_{\alpha\beta}}$ will exist for every α and β. We show that

$$\lim_{\alpha,\beta=-\infty} (P)\int_{E_{\alpha\beta}} = (P)\int_E$$

by using the property (2) of § 6. On the other hand if $(P)\int_{E_{\alpha\beta}}$ exists for every α, β, then we can show that the $E_{\alpha\beta}$ are measurable relative to E, and to each other by using property (3) of § 5, from which we conclude that f is measurable relative to E. The existence of

$$(P)\int_E = \lim_{\alpha,\beta} \int_{E_{\alpha\beta}},$$

follows from property (2) of § 6.

II. THE BOREL AND DENJOY DEFINITIONS OF INTEGRATION.

Suppose that, in an interval (a, b), $f(x)$ is not bounded in every vicinity of a point c; suppose further that the definite (Riemann) integral exists in the intervals $(a, c-\epsilon')$ and $(c+\epsilon'', b)$ for every ϵ' and ϵ''. Then consistent with the continuity of the definite integral as a function of its limits we define (as suggested by Cauchy)

$$\int_a^b f(x) = \lim_{\epsilon'=0} \int_a^{c-\epsilon'} f(x) + \lim_{\epsilon''=0} \int_{c+\epsilon''}^b f(x).$$

Evidently the extension of this to the case in which the number of points of (a, b) in every vicinity of which $f(x)$ is not bounded is finite, is immediate.

When the number of points in the vicinity of which $f(x)$ is not bounded becomes infinite then there are two types of definition. One of them gives a definition by means of a single limiting process, the other by a denumerable set of such processes. The first of these leads to the Harnack-Jordan-Moore and Borel types of integration, the other to the Dirichlet, extended by Hoelder and Lebesgue, and Denjoy definitions of integration.

1. *The Harnack-Jordan Definition.*—(Cf. Harnack (9), pages 220 ff.; Jordan (12), pages 50, 51; Moore (15); Nalli (16), pages 20–27.) We note first of all that the set of points in every vicinity of which $f(x)$ is not bounded constitute a closed set. We call this set for the present the set of singularities, and denote it by Z. Then Harnack defines the integral of $f(x)$ as follows:

Suppose the set Z of singularities of $f(x)$ is of content zero. Enclose them in a finite set of intervals of total length ϵ. Let $f_1(x)$ be zero in the interior of the enclosing intervals, and equal to $f(x)$ everywhere else and suppose that $\int_a^b f_1(x)dx$ exists. If this integral approaches a finite limit as ϵ approaches zero, this limit is said to be the integral of $f(x)$ from a to b.

Jordan gives a definition which is equivalent to this in case the content of Z is zero. His definition is as follows:

Divide (a, b) into any finite number of intervals of maximum length δ. Exclude the intervals containing points of the set Z, and suppose that the (Riemann) integrals of $f(x)$ exist on the remaining intervals. If the sum of these integrals approaches a definite finite limit when δ approaches zero, this is defined to be the integral of $f(x)$ from a to b.

Moore ((15), pages 300–302) observes that the Harnack definition can be applied when the set Z is replaced by another Z_0 containing it, and that the resulting integral is in reality a function of the set Z_0. Further, that in case the set of singularities Z is non-existent then the integral of $f(x)$ on the basis of the set Z_0 is equal to the ordinary integral of $f(x)$ if and only if the set Z_0 is of content zero. For that reason it is desirable to restrict the consideration of these integrals to sets Z of content zero, and it is to be supposed that Jordan assumed this to be the case even though he does not state this fact explicitly.*

2. *The Borel Integral.*—(Cf. Borel (2*a*), pages 199–205; (2*b*), pages 249–252; Hahn (8).) Borel has given a definition similar to that of Harnack except that he assumes that the set of singularities Z is of measure zero, and may be everywhere dense, i. e., in general not closed. In this case Z is not necessarily the set of all points in every neighborhood of which the function $f(x)$ becomes unbounded.

* No change, however, is made in the definition in the 3d edition of his Cours d'Analyse.

Before taking up the definition which Borel gives, it may be desirable to observe that it is possible to define a Riemann integral of a function $f(x)$ on a perfect or closed set P. Viz., if in the sum

$$\sum_{i=1}^{n} f(\xi_i)(x_i - x_{i-1})$$

we replace the length of the interval (x_i, x_{i-1}) by the content h_i of the part of P contained in the interval, and choose ξ_i to belong to the ith interval and P, then the limit of this sum as the length of the maximum subinterval approaches zero is the Riemann integral of $f(x)$ on P. We note that a necessary and sufficient condition for the existence of a Riemann integral on P is that the set of points of P at which $f(x)$ is discontinuous of measure be zero.

Borel's definition is then equivalent to the following

DEFINITION. *Suppose the set Z of singularities is of measure zero, and such that if it is enclosed in a set of intervals of total length ϵ, each interval containing at least one point of the set Z, then the Riemann integral of $f(x)$ exists on the complementary set P_ϵ. If*

$$\lim_{\epsilon=0} \int_{P_\epsilon} f(x)$$

exists, for all possible enclosures of the type described, it is defined to be the (B) integral of $f(x)$ on (a, b).

Borel assumes that this definition is to be applied to functions which are not bounded on (a, b), but it can obviously be applied also to functions which are bounded. The relation of this definition to the one of Lebesgue has been discussed by Hahn (8). He proves the following theorems:

(1) If $f(x)$ is absolutely (B) integrable, i. e., if $(B)\int |f(x)|\,dx$ exists, then $f(x)$ is also Lebesgue integrable.

Apparently this includes as a special case that in which $f(x)$ is a bounded function.

By way of proof we note that $(R)\int_{P_\epsilon} = (L)\int_{P_\epsilon}$ for any f for which the first integral exists. Let now $\epsilon_1, \cdots, \epsilon_n, \cdots$, be such that $\lim_n \epsilon_n = 0$, and let $P_n = P_{\epsilon_n}$ be the corresponding closed sets of measure greater than $b - a - \epsilon_n$, which are chosen in such a way that P_n contains P_{n-1}. If then $f_n = f$ on P_n and

zero everywhere else, then the $|f_n|$ will form a monotonic non-decreasing sequence of functions approaching $|f|$ except possibly at a set of measure zero. Hence by the properties of (L) integrals

$$(B)\int_a^b |f| = \lim_n \int_{P_n} |f| = \lim_n (L)\int_{P_n} |f|$$
$$= \lim_n (L)\int_a^b |f_n| = (L)\int_a^b |f|.$$

From this we conclude the existence of $(L)\int_a^b f$ and since meas P_ϵ approaches $b - a$, its equality to $(B)\int_a^b f$.

(2) A necessary and sufficient condition under the hypothesis of the definition that f be (B) integrable is that (a) f be (L) integrable, i. e., summable, on the set $Z + Z'$, where Z' is the derived set of Z; and (b), if $u_n = (a_n, b_n)$ are the intervals complementary to the closed set $Z + Z'$, and $W(u_n)$ is the maximum value of $\left|\int_{u_n'} f\right|$ for all intervals u_n' interior to u_n, then $\Sigma_n W(u_n)$ shall form a convergent series. Moreover

$$(B)\int_a^b f(x)dx = \Sigma_n \int_{u_n} f(x)dx + (L)\int_{Z+Z'} f(x)dx,$$

where

$$\int_{u_n} = \lim_{u_n' = u_n} \int_{u_n'}.$$

We refer to the original memoir of Hahn for the proof of this theorem. In case the set $Z + Z'$ is of measure zero, and therefore of content zero, the Borel definition may be reduced to that of Harnack-Moore and the above equality lacks the second term on the right, in which form it was originally derived by Moore (cf. (15), pages 324 ff.). We shall refer to this theorem as the Moore-Hahn theorem.

(3) *The Borel definition is not equivalent to the Lebesgue definition of integration, in particular there exist functions which are Borel integrable without being Lebesgue integrable, and functions which are Lebesgue integrable without being Borel integrable.*

For, as given, the definition of Borel includes functions which are not absolutely integrable (cf. Moore (15), page 327), while any Lebesgue integrable function is absolutely integrable. On the other hand the function $f(x)$, which is zero except at the points of a non-dense perfect set of measure greater than zero, at which it has the value 1, though (L) integrable, is not (B) integrable. For Hahn has shown that there does not exist any set of measure zero such that for every enclosure of this set, each interval of the enclosure containing a point of this set, the function $f(x)$ is Riemann integrable on the set of points complementary to the intervals.

We can give a definition which will include as special cases the Lebesgue and Borel integrals by assuming that $f(x)$ is bounded and Lebesgue integrable on the sets P_ϵ and that the integral on the interval is the limit of the integrals on the sets P_ϵ. Then a theorem similar to the Moore-Hahn theorem will hold, the integrals $\int_{u_{n'}}$ being now taken in the sense of Lebesgue.

A further extension is to assume that on the sets P_ϵ, $f(x)$ is not necessarily bounded, but only Lebesgue summable. That this actually is an extension can be shown by a simple example.

Suppose that the interval is (0, 1) and $f(x)$ is defined as follows: (a) in the interval $i_n = \left(\frac{1}{n}, \frac{1}{n+1}\right)$, $f(x) > 0$ if n is even and $f(x) < 0$ if n is odd;

$$(b) \qquad \left|\int_{i_n} f(x)dx\right| = \frac{1}{n};$$

(c) $f(x)$ is continuous throughout i_n except that

$$\lim_{\epsilon \doteq 0} \left| f\left(\frac{1}{n} + \epsilon\right)\right| = \infty.$$

Then it can be shown* that the Borel integral of this function in the interval (0, 1) does not exist, the set of singularities being the points $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots, 0$; if, however, we apply the last definition suggested above with the set of singularities $Z = (0)$, we obtain an integral.

* Cf., for instance, Moore (15), p. 329.

We note finally that the Harnack-Borel integrals do not have the property which we find in the case of Lebesgue and Riemann integrals that if f and g are integrable, then $f+g$ is also integrable. Borel has suggested that in such a case we set by definition

$$\int (f+g) = \int f + \int g.$$

3. *The* (B^*) *Integral.*—(Cf. Hahn (8), page 9; Nalli (16), pages 84–97.) In the Borel definition of integral we note (*a*) that the set of singularities Z is a fixed set of points (i. e., in a way the integral is a function of the set Z); (*b*) every allowable set of intervals is such that each interval of the set contains at least one point of the set Z. A definition has been suggested* in which both of these requirements are dropped. It is as follows:

DEFINITION. *Suppose* (*a*) $f(x)$ *is such that for every* $\epsilon > 0$, *there exists a perfect set* P_ϵ *of measure greater than* $b-a-\epsilon$, *for which the Riemann integral* $\int_{P_\epsilon} f$ *exists; further suppose* (*b*) *that*

$$\lim_{\epsilon=0} \int_{P_\epsilon} \overline{f}$$

exists. Then this limit is said to be the (B^*) *integral of* $f(x)$ *on* (a, b).

Obviously since for every ϵ we can find in any measurable set E a perfect set P_ϵ such that meas E − meas $P_\epsilon < \epsilon$ we can apply the same type of definition to the definition of an integral extended over a measurable set E.

We note the following propositions:

(1) A necessary and sufficient condition that there exist a (B^*) integral is that for every $\epsilon > 0$ there exists a δ_ϵ such that if P_1 and P_2 have measures greater than $b-a-\delta_\epsilon$, and $\int_{P_1}$ and $\int_{P_2}$ exist, then

$$\left| \int_{P_1} - \int_{P_2} \right| \leqq \epsilon.$$

(2) If the (B^*) integral of $f(x)$ exists on (a, b), then for any sequence of ϵ's such that $\epsilon_1 \geqq \epsilon_2 \geqq \cdots$ and $\lim_n \epsilon_n = 0$

* Cf. Hahn (8), p. 9. Nalli (16), p. 85, erroneously asserts that Borel's definition is the one which we now give.

we can find a sequence of perfect sets P_n of measure greater than $b - a - \epsilon_n$, each containing the preceding, such that $\int_{P_n}$ exists and $\lim_n \int_{P_n} = (B^*) \int_a^b$.

For this purpose we obtain a series of P_ϵ: $P_{\epsilon_1 - \epsilon_2}$, $P_{\epsilon_2 - \epsilon_3}$, $\cdots$. For P_n we take the greatest perfect set common to the sets $P_{\epsilon_m - \epsilon_{m+1}}$, for $m \geqq n$. This will evidently have measure greater than $b - a - \epsilon_n$, and be contained in P_{n+1}.

(3) If $f(x)$ is bounded in (a, b) and fulfils condition (a) of the definition, then the (B^*) integral exists.

(4) If f and g are (B^*) integrable, then $f + g$ is (B^*) integrable and we have

$$(B^*) \int (f + g) = (B^*) \int f + (B^*) \int g,$$

a proposition which does not hold for the (B) integral.

(5) *If f is integrable according to the Lebesgue definition, then f is also (B^*) integrable, the values of the integrals being the same.*

For if f is (L) integrable, then f is measurable. From the theorem of Lusin* on measurable functions:

If f is measurable in an interval (a, b), then for every $\epsilon > 0$ there exists a non-dense perfect set P_ϵ of measure greater than $b - a - \epsilon$ on which f is continuous,

it follows that the condition (a) of the definition is satisfied, i. e., for every $\epsilon > 0$ there exists a P_ϵ of measure greater than $b - a - \epsilon$ for which the $(R) \int_{P_\epsilon}$ exists. Evidently

$$(R) \int_{P_\epsilon} = (L) \int_{P_\epsilon}.$$

Moreover we know that the (L) integral is absolutely continuous, i. e., if $\lim_n \text{meas } E_n = \text{meas } E$, then $\lim_n (L) \int_{E_n} = (L) \int_E$. Hence since

$$\lim_{\epsilon = 0} \text{meas } P_\epsilon = b - a,$$

it follows that

* Cf. *Comptes Rendus*, vol. 154 (1912), p. 1689. Cf. also Lebesgue, *Comptes Rendus*, vol. 137 (1903), p. 1229.

$$\lim_{\epsilon=0} (R)\int_{P_\epsilon} = \lim_{\epsilon=0} (L)\int_{P_\epsilon} = (L)\int_a^b,$$

i. e., the (B^*) integral exists and is equal to the (L) integral. The same method of reasoning would show that if $(L)\int_E$ exists, where E is any measurable set, then $(B^*)\int_E$ exists also.

We note in passing that the Hahn example of a function not (B) integrable is obviously (B^*) integrable.

On the other hand we also have

(6) *If $f(x)$ is (B^*) integrable on (a, b), then it is also (L) integrable.*

We observe that if we show that an f which is (B^*) integrable is absolutely (B^*) integrable, i. e., $(B^*)\int |f|$ exists, we can apply the method of proof used for (1) of § 2 above, for the case of (B) integrability. We therefore proceed to show

(7) If $f(x)$ is (B^*) integrable then $|f|$ is also (B^*) integrable.

Suppose if possible that $|f|$ is not (B^*) integrable. Then from (2) above it would follow that $(B^*)\int |f|$ is infinite. Hence if $f_1 = \frac{1}{2}(|f| + f)$ and $f_2 = \frac{1}{2}(|f| - f)$, then by (4) $(B^*)\int f_1$ and $(B^*)\int f_2$ are both infinite. For every M we can then determine a perfect set $P_{1\epsilon}$ of measure greater than $b - a - \epsilon$ such that $\int_{P_{1\epsilon}} f_1 > M$. For the same ϵ we can determine a perfect set $P_{2\epsilon}$ on which $(R)\int_{P_{2\epsilon}} f_2$ exists, and consequently f_2 is bounded on $P_{2\epsilon}$. Suppose $f_2 < m$ on $P_{2\epsilon}$. Then $(R)\int_{P_{2\epsilon}} f_2 < m(b - a)$. Let the set E be the sum of the measurable sets: (a) the subset of $P_{1\epsilon}$ for which $f_1 > 0$ and hence $f_2 = 0$; (b) the subset of $P_{2\epsilon}$ for which $f_2 > 0$; (c) the greatest common subset of $P_{1\epsilon}$ and $P_{2\epsilon}$. Then the measure of E will be greater than $b - a - 2\epsilon$ and

$$(L)\int_E f = (R)\int_{P_{1\epsilon}} f_1 - (R)\int_{P_{2\epsilon}} f_2 > M - m(b - a).$$

However, by (5) we can find a perfect set P_ϵ of measure greater

than or equal to $b - a - 2\epsilon$, contained in E, such that the difference between $(L)\int_E f$ and $(R)\int_{P_\epsilon} f$ is as small as we please, i. e., such that

$$(R)\int_{P_\epsilon} f \geqq M - m(b - a).$$

Since M has been chosen arbitrarily, it follows that

$$\lim_{\epsilon=0} \int_{P_\epsilon} f$$

does not exist, i. e., f is not (B^*) integrable, which is a contradiction. Hence the theorem.

We thus have the result

(8) (B^*) *integrability is equivalent to* (L) *integrability, the values of the integrals being the same.*

The absolute convergence of the (B^*) integral is another instance of a phenomenon observed by Stolz* and Moore.† In the (B^*) integration while the singularities are enclosed in a set of intervals of total length less than ϵ, it is not required that every interval contain a point of the set, i. e., we have a *broad* enclosure. On the other hand, in the (B) integration, every interval must include a point of the set of singularities, i. e., we have a *narrow* enclosure. The broad enclosures yield absolutely convergent integrals, the narrow enclosures may lead to non-absolutely convergent integrals.

4. *The Dirichlet-Lebesgue Extension of the Cauchy Definition.*—(Cf. Lebesgue (13), pages 9–14.) The Cauchy definition of integral of a function $f(x)$, which has only a finite number of points in every neighborhood of which $f(x)$ is not bounded, can be extended so as to give the Harnack and Borel definitions of integration as a result of a single limiting process. We turn now to the other type of definition which requires a finite or denumerable infinitude of limiting processes.

The first step in this direction was taken by Dirichlet, who extends the Cauchy definition to the case in which the interval (a, b) has a set of singularities Z whose derived set Z' contains a finite number of points $x_1, \cdots, x_n$. For in any subinterval interior to the interval (x_{i-1}, x_i) we have only a finite number of

* *Wiener Berichte*, vol. 108_{2a} (1899), p. 1235.
† (5), pp. 302–304, 323.

singularities. To find $\int_{x_{i-1}-\epsilon}^{x_i+\epsilon'} f(x)dx$, for any two positive ϵ and ϵ', we can then apply the Cauchy definition. If the limit of this integral exists as ϵ and ϵ' approach zero, it is the integral in the interval (x_{i-1}, x_i), and the integral in (a, b) is the sum of these integrals.

We observe that the Cauchy-Dirichlet definitional process makes use of the following two conditions:

(a) If an integral exists on each of a finite set of consecutive intervals (a_1, a_2); (a_2, a_3); $\cdots$; (a_{n-1}, a_n), then

$$\int_{a_1}^{a_n} = \sum_{i=1}^{n-1} \int_{a_i}^{a_{i+1}}.$$

(b) If $\int_{a+\epsilon'}^{b-\epsilon''}$ exists for every ϵ' and ϵ'', and if this integral approaches a limit as ϵ' and ϵ'' approach zero, this limit is defined to be $\int_a^b$.

Evidently the Dirichlet manner of reasoning is immediately extensible to the case in which Z'' and then $Z^{(\alpha)}$, α being any finite or transfinite number of the first and second classes, contains a finite number of points,* i. e., if Z is a closed reducible set, then this method will give a definition of integration over the interval (a, b). Lebesgue† shows that this same result may be attained by the following definition:

A function $f(x)$ has an integral in the finite interval (a, b) if there exists a continuous function $F(x)$, unique except for the addition of a constant, such that we have in every interval (α, β) where $f(x)$ is continuous

$$\int_\alpha^\beta f(x)dx = F(\beta) - F(\alpha).$$

$F(x)$ is the indefinite integral of $f(x)$ and we have

$$\int_a^b f(x)dx = F(b) - F(a).$$

In applying the process just discussed we start from $Z^{(\alpha)}$, and work back to Z by a denumerable set of limiting proc-

* Cf. Schoenfliess, Bericht über die Mengenlehre, I (1900), p. 185.

† Cf. (13), pp. 10–14. See also Hölder, *Math. Annalen*, vol. 24 (1884), pp. 90 ff.

esses, in every interval into which $Z^{(\alpha)}$ divides (a, b). We can obtain the same result in case Z is reducible, by starting with Z and working up to $Z^{(\alpha)}$. For consider the intervals complementary to Z'. By the two conditions of the Cauchy-Dirichlet process, we can then define an integral in each of these intervals. Evidently the same holds true of the intervals complementary to Z'', and so on. By a well-known theorem in the theory of point sets* we arrive finally at the intervals complementary to a set $Z^{(\alpha)}$ which is either zero or perfect. If it is zero, then we have an integral for the entire interval. If perfect, then we have obtained an integral in each interval complementary to this perfect set $Z^{(\alpha)} = P$. If $W(u_n)$ is the maximum absolute oscillation of $\int_{u_n'}$, u_n' being any interval interior to u_n, and $\Sigma W(u_n)$ is convergent, and P is of content zero, then the Moore-Hahn theorem (2) of § 2 would suggest that we define

$$\int_a^b f(x)dx = \sum_{n=1}^{\infty} \int_{u_n} f(x)dx.$$

We observe that this will include the Harnack integral as a special case.†

We might proceed to a discussion of the case in which the content of P is not zero, or the series of oscillations or integrals over the complementary intervals u_n is not absolutely convergent, but we prefer to take up briefly the Denjoy definition of integration which combines the two definitional conditions of Cauchy-Dirichlet with the Moore-Hahn theorem, and includes as special cases almost all of the definitions so far considered in this chapter.

5. *The Denjoy Definition of Integration.*—(Cf. Denjoy (3); Nalli (16), pages 122 ff.) We give first a few definitions.

(*a*) A function $f(x)$ is said to be *not summable at a point* x_0 of an interval, if $f(x)$ is not summable in every interval containing the point x_0. Similarly $f(x)$ is not summable at a point x_0 on a perfect set P, if the function $\varphi = f$ on P and zero at points not on P is not summable at x_0. In either case the points at which $f(x)$ is not summable form a closed set.

(*b*) Let $[u_n]$ be the set of intervals complementary to a perfect set P, relative to an interval (a, b), and suppose that

* Cf. Hobson, Theory of Functions, pp. 92, 93.
† Cf. Moore (15), p. 330.

for every n there is defined a constant $W(u_n)$. Then the series $\Sigma W(u_n)$ is said to be *absolutely convergent on an interval* δ if the part of the series $\Sigma W(u_n)$ which corresponds to the intervals interior to δ is absolutely convergent. It is *absolutely convergent at a point* x_0 of the perfect set P if there exists an interval containing x_0 in which $\Sigma_n W(u_n)$ is absolutely convergent. Evidently the points of P at which $\Sigma_n W(u_n)$ is not absolutely convergent form a closed set.

Then Denjoy defines an integral $V(a, b)$ on the interval (a, b) as follows:

(*A*) The integral of $f(x)$ in an interval in which $f(x)$ is summable is the Lebesgue integral.

(*B*) If the integral has been defined for a finite number of consecutive intervals: (a_1, a_2); (a_2, a_3); $\cdots$; (a_{n-1}, a_n) then $V(a_1, a_n)$ is defined to be

$$V(a_1, a_n) = \sum_{i=1}^{n-1} V(a_i, a_{i+1}).$$

(*C*) If (*a*) $f(x)$ is summable on a perfect set P, whose complementary intervals relative to an interval (c, d) are u_n; (*b*) $V(u_n')$ has been defined on every interval u_n' containing no points of P in its interior and $W(u_n)$ is the least upper bound of $|V(u_n')|$ for all possible u_n' interior to u_n; and (*c*) $\Sigma_n W(u_n)$ is convergent, then

$$V(c, d) = \Sigma_n V(u_n) + \int_P f(x)dx.$$

The function $f(x)$ is said to be *Denjoy* (*or* (*Dn*)) *integrable*, if the following conditions are fulfilled:

I. The set of points of any perfect set P at which $f(x)$ is not summable on P is not dense on P.

II. If $V(c', d')$ has been defined for every interval (c', d') interior to $V(c, d)$, then

$$\lim_{c \doteq c',\ d \doteq d'} V(c', d')$$

exists and is defined to be $V(c, d)$.

III. The set of points of any perfect set P at which the series of maximum oscillations $W(u_n)$ of V on the intervals complementary to P is not absolutely convergent is not dense on P.

The proof that these three definitional conditions and the

three integrability conditions suffice to determine for a function $f(x)$ a finite number, its integral, is rather long. We mention only the chief features of the proof.

A central position is occupied by the theorem on point sets: If we have a series of closed sets $E^{(\alpha)}$, corresponding to the transfinite numbers of the first and second class, each contained in the preceding and not dense on it, then there exists a transfinite number α with a precedent, such that $E^{(\alpha-1)} \neq 0$ and $E^{(\alpha)} = 0$.

Condition I when applied to the interval (a, b) gives a closed set E_1, not dense on (a, b) at every point of which f is not summable on (a, b). We break up E_1 into $P_1 + R_1$, where P_1 is perfect and R_1 is reducible. Then we replace (a, b) by P_1 and apply condition I again. This yields $E_2 = P_2 + R_2$. And so we continue. The theorem just given shows that this process must terminate after a denumerably infinite number of steps.

Condition II, with the definitional condition B, when applied to the intervals complementary to E_1 gives a value for the intervals complementary to the derived set E_1' of E_1. The same two conditions lead us to the intervals complementary to E_1'' and eventually to the intervals complementary to P_1.

In the same way, definitional condition C together with the conditions II and III give us a value for $V(g, h)$ in any interval (g, h) which contains points of P_1 but not of E_2, i. e., in any interval complementary to E_2. Similarly we find the value of $V(g, h)$ for every interval complementary to $E_{\alpha+1}$, the values in the intervals complementary to P_α having been determined.

We thus obtain a value for the integral $V(a, b)$ in the interval (a, b) after a denumerably infinite number of steps.

We note the following properties:

(1) If $f(x)$ is such that $|f(x)|$ is Denjoy integrable, then $f(x)$ is Lebesgue integrable.

For evidently if $f(x) \geqq 0$, and there existed a point at which $f(x)$ were not summable on (a, b), then $V(a, b)$ would also be infinite.

(2) From the Moore-Hahn theorem of § 2 it follows that if $f(x)$ is (B) integrable it is also Denjoy integrable, the definitional condition C being an immediate extension of the expression for the (B) integral. The same result holds true

if in the (B) integral we replace the Riemann integrals by Lebesgue integrals. Just how the definitions compare when Riemann integrability on P_ϵ is replaced by Lebesgue summability, we do not know, except that there are functions which are Denjoy integrable without being (B) integrable in any extensions of the definition, an immediate consequence of

(3) If f and g are Denjoy integrable, then $f+g$ is also Denjoy integrable, and we have

$$(Dn)\int f + (Dn)\int g = (Dn)\int (f+g).$$

We have further

(4) The indefinite Denjoy integral $V(a, x)$ is a continuous function and has as derivative $f(x)$ except for a set of measure zero.

(5) If the continuous function $F(x)$ has at every point a finite derivative $f(x)$ then this derivative is Denjoy integrable, and the Denjoy integral of $f(x)$ has the value $F(x) - F(a)$.

This result is of importance in the theory of the derivatives of continuous functions. In the case in which the derivative is finite at every point, it gives a means of returning from the derivative to the original function. For the Riemann integral, the derived function $f(x)$ must be Riemann integrable; for the Lebesgue integral, $f(x)$ must be summable, and this carries with it that $F(x)$ be of bounded variation; but $f(x)$, finite at every point, is Denjoy integrable.*

For the proofs of these theorems we refer to the Denjoy articles, as well as the Nalli thesis, which contains a careful analysis of the Denjoy first notes on the subject.

Quite recently Khintchine† has pointed out that in order to obtain an integral by the Denjoy method of definition, it is sufficient to replace the condition of convergence of the maximum oscillations $W(u_n)$ in the intervals u_n, in the definitional condition C and the integrability condition III, by the weaker condition of convergence of the series of variations of the integrals $\Sigma_n V(u_n)$, in as much as $\Sigma_n V(u_n)$ and not $\Sigma_n W(u_n)$

* Cf. in this connection Lusin: *Comptes Rendus*, volume 155 (1912), pp. 1475–7. See also *Annali di Matematica*, ser. 3, vol. 26 (1917), pp. 77–131. Pp. 118–131 contain a very interesting suggestion for extending the definition of integration by the use of Fourier series.

† "Sur une extension de l'intégrale de M. Denjoy," *Comptes Rendus*, vol. 162 (1916), pp. 287–291. Cf. also pp. 377–380.

appears in the integration expression. However, for an integral defined in this way it is no longer true that the derivative of the indefinite integral $V(a, x)$ is the function $f(x)$ except for a set of measure zero. By defining the term "integrals approaching zero asymptotically at a point x_0 of a perfect set P" to mean that if d_n is the distance between x_0 and the nearest extremity of the interval u_n complementary to P, then

$$\lim_n \frac{W(u_n)}{d_n} = 0,$$

he shows that a necessary and sufficient condition that the derivative of $V(a, x)$ be $f(x)$ except for a set of measure zero, is that the definitional condition C be replaced by

C'. If f is summable on P in an interval (c, d), if, moreover, the integrals approach zero asymptotically except for a set of measure zero on P, then

$$V(c, d) = (L)\int_P f + \Sigma_n V(u_n);$$

and the integrability condition III by

III'. For every perfect set P, there exists a portion* π of P such that if $\delta_1, \cdots, \delta_n, \cdots$ are the intervals complementary to π, the series $\Sigma_n V(\delta_n)$ converges and the integrals approach zero asymptotically on π except for a set of measure zero on π.

Both of the Khintchine suggestions give actual extensions of the Denjoy definition, i. e., give an integral for a function not integrable by the definition as given by Denjoy.

* Cf., for instance, Denjoy (4(a)), p. 120.

(*To be continued.*)

A CONTINUOUS FUNCTION WHOSE DEVELOPMENT IN BESSEL'S FUNCTIONS IS NON-SUMMABLE OF CERTAIN ORDERS.

BY PROFESSOR CHARLES N. MOORE.

(Read before the American Mathematical Society, September 4, 1916.)

THE first example of a continuous function whose Fourier's development diverges at one or more points was given, as is well known, by Du Bois-Reymond.* Much more recently Haar† showed how to construct a continuous function whose Sturm-Liouville development is divergent, and a continuous function whose development in Legendre's functions has this same property. Somewhat later Gronwall‡ gave in explicit form a function whose development in Legendre's functions is not summable (C_k) for $0 \leqq k \leqq \frac{1}{2}$, at the point $x = 1$.

As far as the writer is aware, no examples have thus far been given of continuous functions whose developments in Bessel's functions diverge or fail to be summable of certain orders at one or more points. In the present paper there is exhibited in explicit form a continuous function whose development in Bessel's functions of order zero is not summable (C_k) for $0 \leqq k < \frac{1}{2}$, at the point $x = 0$. This function is analogous to a function given by Fejér§ whose Fourier development diverges for $x = 0$. The proof of the non-summability, however, follows different lines from Fejér's proof of the divergence for his example.

If we represent by A_r the rth coefficient in the development of an arbitrary function $f(x)$ in terms of Bessel's functions, we have

$$A_r = \frac{\int_0^1 x f(x) J_0(\lambda_r x) dx}{\int_0^1 x J_0^2(\lambda_r x) dx}, \tag{1}$$

where λ_r is the rth root of the equation

* *Abhandlungen der Bayerischen Akademie der Wissenschaften*, vol. 12 (1876).

† Inaugural Dissertation, Göttingen (1909); also *Math. Annalen*, vol. 69 (1910).

‡ *Math. Annalen*, vol. 75 (1914).

§ *Journal für die reine und angewandte Mathematik*, vol. 137 (1910).

$$(2) \qquad l\lambda J_0'(\lambda) + hJ_0(\lambda) = 0,$$

l and h being any constants not both zero. Moreover, we have for λ_r

$$(3) \qquad \lambda_r = (r+q)\pi + \frac{\psi(r)}{r},$$

where q is a constant and $\psi(r)$ is used to represent a function of r that remains finite for all values of r.

Consider now the function $f(x)$ defined by the following equation

$$(4) \qquad f(x) = \sum_{n=1}^{\infty} \frac{\sqrt{x}\cos(2^n+q)\pi x}{n^2},$$

where q is the q of (3). We will show that for the coefficients A_r in the development of this function in terms of Bessel's functions

$$(5) \qquad \lim_{m\to\infty} \frac{A_{2^m}}{(2^m)^k} = \infty \qquad (0 \leqq k < \tfrac{1}{2}).$$

Thus it will follow that the series

$$(6) \qquad A_1 + A_2 + A_3 + \cdots + A_r + \cdots$$

is not summable (Ck) for $0 \leqq k < \frac{1}{2}$, since for any series so summable the limit on the left-hand side of (5) exists and is equal to zero.* But since $J_0(0) = 1$, the development of $f(x)$ in Bessel's functions of order zero reduces to the form (6) for $x = 0$, and therefore it will have been shown that this development is not summable (Ck) for $0 \leqq k < \frac{1}{2}$ at the point $x = 0$

It only remains then to establish the relationship (5). Since the series on the right-hand side of (4) is uniformly convergent in the interval $0 \leqq x \leqq 1$, we have from (1)

$$(7) \qquad A_r = \frac{1}{\int_0^1 xJ_0^2(\lambda_r x)dx} \sum_{n=1}^{\infty} \frac{\int_0^1 x^{\frac{1}{2}}\cos(2^n+q)\pi x J_0(\lambda_r x)dx}{n^2}.$$

From the asymptotic expansion of $J_0(x)$ we have

$$(8) \qquad J_0(\lambda x) = \sqrt{\frac{2}{\pi\lambda x}}\cos\left(\lambda x - \frac{\pi}{4}\right) + \frac{\psi(\lambda x)}{(\lambda x)^{\frac{1}{2}}}.$$

* Cf. Chapman, *Proc. London Math. Society*, ser. 2, vol. 9 (1911), p. 379.

Making use of (8), we see that the general term of the series on the right-hand side of (7) may be written in the form

$$\frac{1}{n^2}\sqrt{\frac{2}{\pi\lambda_r}}\int_0^1 x\cos\left(\lambda_r x-\frac{\pi}{4}\right)\cos(2^n+q)\pi x dx$$

(9)

$$+\frac{1}{n^2\lambda_r^{\frac{1}{2}}}\int_0^1\cos(2^n+q)\pi x\psi(\lambda_r x)dx.$$

If now we set $r=2^m$ and make use of the relation (3), it is readily seen that the sum of the series whose general term is the second term of (9) is of the form

$$(10)\qquad O\left(\frac{1}{\lambda_r^{\frac{1}{2}}}\right)=O\left(\frac{1}{\sqrt{\lambda_r}}\cdot\frac{1}{2^m}\right)=o\left(\frac{1}{\sqrt{\lambda_r}}\cdot\frac{1}{m^2}\right),$$

where we have used $O(x)$ to represent any function which after division by x remains finite as x becomes infinite, and $o(x)$ to represent any function which after division by x approaches zero as x becomes infinite.*

We have further to consider the series whose general term is the first term of (9). Making use of (3) we may write

$$\cos\left(\lambda_r x-\frac{\pi}{4}\right)$$

$$(11)\qquad =\cos\frac{\pi}{4}\cos(r+q)\pi x+\sin\frac{\pi}{4}\sin(r+q)\pi x$$

$$+\cos\left\{(r+q)\pi x-\frac{\pi}{4}+\frac{\psi(r)}{r}x\right\}-\cos\left\{(r+q)\pi x-\frac{\pi}{4}\right\},$$

and the last two terms on the right-hand side of (11) may be written in the form

$$(12)\qquad -\frac{\psi(r)}{r}x\sin\left\{(r+q)\pi x-\frac{\pi}{4}+\theta\right\}\quad\left(|\theta|<\left|\frac{\psi(r)}{r}x\right|\right).$$

Substituting from (11) in the first term of (9) and making use of (12), we reduce this term to the form

$$\frac{1}{n^2\sqrt{\pi\lambda_r}}\int_0^1 x\cos(2^m+q)\pi x\cos(2^n+q)\pi x dx$$

* The use of O in this sense is due to Bachmann. Cf. Analytische Zahlentheorie, vol. 2, p. 401. The corresponding use of o is due to Landau. Cf. Handbuch der Lehre von der Verteilung der Primzahlen, vol. 1, p. 61.

$$+\frac{1}{n^2\sqrt{\pi\lambda_r}}\int_0^1 x\sin(2^m+q)\pi x\cos(2^n+q)\pi x dx$$

(13)

$$-\frac{\psi(2^m)}{2^m\cdot n^2}\cdot\sqrt{\frac{2}{\pi\lambda_r}}\int_0^1 x^2\sin\left\{(2^m+q)\pi x - \frac{\pi}{4}+\theta\right\}\cos(2^n x+q)\pi x dx.$$

The last term in (13) is obviously the general term of a series whose sum is of the form (10). Consider next the first term in (13). If $n \neq m$, the integral in this term reduces to the form

$$\frac{\cos(2^m+2^n+2q)\pi-1}{2(2^m+2^n+2q)^2\pi^2}+\frac{\sin(2^m+2^n+2q)\pi}{2(2^m+2^n+2q)\pi}$$
$$+\frac{\cos(2^m-2^n)\pi-1}{2(2^m-2^n)^2\pi^2}+\frac{\sin(2^m-2^n)\pi}{2(2^m-2^n)\pi}.$$

This quantity is obviously less in absolute value than $K/|2^m-2^n|$, where K is a positive constant. Moreover, for all values of $n<m$

$$2^m-2^n>2^n(2^{m-n}-1)\geqq 2^n\cdot 2^{m-n-1}=2^{m-1},$$

and for all values of $n>m$

$$2^n-2^m\geqq 2\cdot 2^m-2^m=2^m.$$

Hence it follows that $K/|2^m-2^n|<K/2^{m-1}$, and therefore the sum of all the terms of the form of the first term in (13) for which $n\neq m$ is of the form $O(1/(\sqrt{\lambda_r}\cdot 2^{m-1}))$ and consequently of the form of the last member of (10).

If $n=m$, the integral in the first term of (13) reduces to

$$\frac{\cos 2(2^m+q)\pi-1}{8(2^m+q)^2\pi^2}+\frac{\sin 2(2^m+q)\pi}{4(2^m+q)\pi}+\frac{1}{4},$$

and hence for this value of n the first term of (13) takes the form

$$o\left(\frac{1}{\sqrt{\lambda_r}}\cdot\frac{1}{m^2}\right)+\frac{1}{4\sqrt{\pi}\sqrt{\lambda_r m^2}}. \tag{14}$$

Combining this with the previous result, we see that the series

of which the first term of (13) is the general term has a sum which is of the form (14).

We have finally to discuss the second term of (13). The integral in this term may for $n \neq m$ be reduced to the form

$$\frac{\sin(2^m + 2^n + 2q)\pi}{2(2^m + 2^n + 2q)^2\pi^2} - \frac{\cos(2^m + 2^n + 2q)\pi}{2(2^m + 2^n + 2q)\pi} + \frac{\sin(2^m - 2^n)\pi}{2(2^m - 2^n)^2\pi^2} - \frac{\cos(2^m - 2^n)\pi}{2(2^m - 2^n)}.$$

For $n = m$ it takes the form

$$\frac{\sin 2(2^m + q)\pi}{8(2^m + q)^2\pi^2} - \frac{\cos 2(2^m + q)\pi}{4(2^m + q)\pi}.$$

Hence the series whose general term is the second term of (13) has a sum which is of the form of the last member of (10).

We have now seen that the series whose general terms are the second and third terms of (13), respectively, have sums that are of the form of the last member of (10), whereas the series whose general term is the first term of (13) has a sum that is of the form (14). Hence the series whose general term is (13), or the first term of (9), has a sum which is of the form (14). Since, moreover, the series whose general term is the second term of (9) has a sum which is of the form (10), it follows that the series whose general term is (9), that is, the series in the right-hand member of (7), has a sum which is of the form (14). As the other factor of the right-hand member of (7) is of the form $C\lambda_r + o(\lambda_r)$, where C is a positive constant, it follows that

$$(15) \quad A_r = C_1\frac{\sqrt{\lambda_r}}{m^2} + o\left(\frac{\sqrt{\lambda_r}}{m^2}\right) = C_2\frac{\sqrt{r}}{m^2} + o\left(\frac{\sqrt{r}}{m^2}\right) = C_2\frac{\sqrt{2^m}}{m^2} + o\left(\frac{\sqrt{2^m}}{m^2}\right),$$

where C_1 and C_2 are positive constants and we make use of the fact that we have chosen $r = 2^m$.

It follows from (15) that A_r satisfies the relationship (5), and hence as pointed out before, *the development in Bessel's functions of order zero of the function $f(x)$ defined by* (4) *is not summable* (Ck) *for* $0 \leq k < \frac{1}{2}$ *at the point* $x = 0$.

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FINITE GROUPS.

Theory and Applications of Finite Groups. By G. A. MILLER, H. F. BLICHFELDT, and L. E. DICKSON. New York, Wiley, 1916. 8vo. 17 + 390 pp.

DURING the last half-century the theory of groups has been explored with great vigor and success, and it is now an imposing body of doctrine. Its importance is principally due to the fact that it enters into the very warp and woof of a large number of mathematical textures. "Many branches of mathematics are nothing but the theory of invariants of special groups." This is especially true of geometry, as was made clear once for all by Klein. Geometry is not one science, but an infinite variety of sciences, depending on the choice of the group on which it is based. As the fundamental group becomes larger and more extensive, the corresponding geometry becomes less extensive and more fundamental. Thus, beginning with the elementary geometry of congruent figures, we obtain in succession a number of different geometries among which are the geometry of similar figures, affine geometry, projective geometry, conformal geometry, and analysis situs.

From a purely abstract standpoint groups may be classified in various ways: they are either abelian or non-abelian, finite or infinite, continuous or discontinuous. The principal classes of concrete groups are permutation groups, linear groups, and groups of non-linear transformations, together with numerous geometrical and mechanical applications. Even in the study of point sets and other aggregates groups have been found useful; for instance, in Bernstein's proof that if two cardinal numbers $n\alpha$ and $n\beta$ are equal, then $\alpha = \beta$.

In a linear group the transformations may be homogeneous or fractional, singular or non-singular, but the most striking principle of classification is that which depends on the nature of the coefficients, which may belong to an infinite field, a finite field, or no field whatever. If the coefficients belong to an infinite field or domain, for instance that of all complex numbers, we obtain a most important class of groups, both continuous and discontinuous, with which are closely asso-

ciated groups of collineations and other geometric transformations. If the coefficients belong to a finite (or Galois) field, of which a special case is that in which they are residues with respect to a prime modulus, the groups are themselves necessarily finite. But the coefficients may not belong to any field. Thus they may be residues with respect to a composite modulus, or with respect to a number of different moduli, or with respect to a modular system. Very little has been done toward the investigation of this class of groups.

When we come to groups of non-linear and in particular of non-algebraic transformations, the possible developments are too vast to contemplate and for the most part lie buried in the future. In the case of birational transformations a beginning has been made. The tremendous variety of geometric groups is evident when we recall that the geometry under consideration may be non-euclidean as well as euclidean, that its number of dimensions may be n or even infinite, that the elements which are transformed may be not merely points, but lines, circles, spheres, etc., and finally that the set of such elements, say of points, instead of forming a continuum, may form a point set of the most general type. Since many physical and chemical changes also form groups, Lie was almost justified in his enthusiastic query, "What are the phenomena of the physical and material universe but the transformations of an enormous group, of which the laws of nature are the invariants?"

Even that part of the theory which has arrived at a fair degree of completeness and finish is far too extensive for a single treatise. The volume before us is restricted to finite groups and indeed to certain selected topics in their theory and applications; linear congruence groups are barely touched upon. The book is written by three recognized authorities on various phases of the subject. Considerable original material is included by each of the authors either in the form of new developments or of simplifications in the proofs of the older theorems. The work is strictly up to date; it contains some of the very latest results on the topics treated.

G. A. Miller, who is responsible for Part I, on abstract and permutation groups, is thoroughly at home in this field; indeed, he has contributed so extensively to its theory and has created so many new and valuable phases of it that he may fairly be called one of the world's leading authorities on

the subject. Part II, by Blichfeldt, is devoted to finite groups of linear homogeneous transformations whose coefficients belong to the domain of complex numbers. The writer is eminently fitted for this task, because he is himself responsible for a new and brilliant method of attack, which enabled him to arrive at the first exhaustive enumeration and discussion of the groups in four variables. Part III, by Dickson, covers the Galois theory of equations and applications to certain geometric problems. In this case also the writer is thoroughly qualified for his task. For he is a very productive mathematician in this and various other fields, who has not only given a definitive and apparently final form to the theory of linear congruence groups and thereby added tremendously to our knowledge of simple groups, but has also contributed largely to the progress of mathematics in many other directions.

Naturally the result of this kind of collaboration is a volume consisting practically of three separate treatises, and yet a certain amount of unity has been attained; while Parts II and III are largely independent of one another, they both presuppose a knowledge of some of the more elementary sections of Part I. We already possess treatises and textbooks on finite groups by Jordan and de Séguier in French, by Bianchi in Italian, by Klein, Weber, and Netto in German, and by Burnside, Hilton, and Dickson in English. But this new volume will easily bear comparison with any of them, not only in its wealth of valuable material, much of which is new, but in vigor and directness of approach, in elegance of treatment, and in the thoroughly interesting way in which many of the results are presented. Certain flaws, mostly of a secondary nature, will presently be pointed out, but every reader must admit the general excellence of the work. There is a graceful and becoming dedication to Camille Jordan, "whose fundamental investigations in the theory and applications of finite groups enriched the subject to the extent of converting it into a fundamental branch of mathematics and furnished in a large measure the inspiration for the subsequent great activity in this field."

In Part I, which covers about half the book, the systematic exposition is preceded, in Chapter 1, by a few simple examples of concrete groups, including the groups of rotations of the equilateral triangle and of the square, a trigonometric illustration of the octic group, and a few simple congruence groups

and groups of matrices. In this way the wind is very properly tempered to the shorn lamb, namely, the beginner. Another admirable feature, pedagogically, is the fact that substitution (or permutation) groups are studied before abstract groups. Indeed, one has the feeling, in reading Part I, that every abstract group is clothed in the flesh and blood of a substitution group, and in particular of a regular group.

In Chapter 2 we meet with substitution groups and their properties, including transforms, co-sets, Sylow's theorem, transitive and intransitive groups, primitive and imprimitive groups, the simplicity of the alternating group, the class of a group, the holomorph of a group, and a determination of all the groups of degree $\leqq 5$. Chapter 3 is devoted to the more fundamental theorems on abstract groups, including the cyclic, dihedral, and dicyclic groups, commutator subgroups, ϕ-subgroups, inner isomorphisms, and the representation of a group as a transitive or regular substitution group. In Chapter 4 we come to abelian groups, their invariants, characteristic subgroups, and groups of isomorphisms; also the roots of their elements and the number of elements of a given order; abelian groups which are conformal with non-abelian groups, non-abelian groups in which every subgroup is abelian, and Hamilton groups, in which every subgroup is invariant. The bitter pill of pure abstraction is sugar-coated for us with the delectable group of totitives. Chapter 5 is concerned with prime-power groups, their subgroups of various types, their groups of isomorphisms, and finally a method of constructing all the possible groups of order p^m. A proof is given of the important new theorem that the number of elements in a set of independent generators of a prime-power group is an invariant of the group. In Chapters 6 and 7 the results are more largely due to Miller's own research than in the other chapters, and are of extremely great value and interest. Chapter 6 is devoted to groups having simple abstract definitions and includes the groups generated by two elements having a common square, the groups of the regular polyhedrons, and various generalizations of the latter. Chapter 7 contains an intensive study of groups of isomorphisms. Chapter 8 is concerned with series of composition, chief series, solvable groups, and unsolvable groups of low composite orders.

Now for a few adverse criticisms. The author of Part I does not always get in touch with the beginner; some of the

initial difficulties of the subject are slurred over. For instance, on page 16, in the third sentence, the adverb "evidently" might easily be regarded as ironical by the beginner. Moreover, the fourth sentence on the same page is not true, unless the cycles are assumed to have no letters in common and therefore to be different from the cycles in the product written down in the next paragraph. A number of terms and phrases are not carefully or explicitly defined; e. g., "transitive constituent groups" on page 33, line 25, "(1, m) correspondence" on page 39, line 1, the "holomorph" and "group of isomorphisms" of an abstract group. The fact that a simple isomorphism is a (1, 1) correspondence in which the products of corresponding elements are themselves corresponding elements is nowhere explicitly mentioned; and this is all the more misleading since the term (1, 1) correspondence is used as synonymous with simple isomorphism. In the middle paragraph of page 103 it is implied that a subgroup can be generated by operators not belonging to the subgroup. On pages 34 and 37 no explanation is given as to what is meant by "regarding several substitutions as one substitution."

The following erroneous or ambiguous statements have been noted. On page 70, lines 9, 10, the elements of the commutator are not to be permuted in every *possible* manner, as there stated, but only in such a way that it may remain a commutator. On page 34, line 3 from the bottom, "the symmetric group" should read "two symmetric groups." On page 53, line 3 from the bottom, "of the form $l + mk$" should read "$l + m_1k$ and $l + m_2k$." In § 38, second paragraph, first sentence, the adjective "low" should be inserted before "order," in order to avoid an apparent conflict with page 89, lines 20–26. On page 99, first theorem, last line, "equal to or greater than" should read "divisible by." On page 179, line 5, "α conjugates of H_1" should read "$\alpha - 1$ terms." On page 185, lines 12, 13, "whose order exceeds unity" should read "which is non-invariant." On page 113, last line, the article "the" should be omitted, as it implies that there is only one Sylow subgroup of each given order. The first sentence on page 174 contains the erroneous statement that the subgroups there mentioned are invariant; but the error is corrected by implication in the next paragraph.

Part II, consisting of Chapters 9–13, gives us a more direct approach to the finite groups of linear transformations,

especially those in three variables, than is contained in any earlier text. We start off, in Chapter 9, along a well-beaten track, which takes us to the characteristic equation, Hermitian invariants, unitary transformations, reducible or intransitive groups, the canonical form, etc. In Chapter 10 we encounter the groups in two variables, which are determined by Klein's original geometric method, somewhat simplified. Thus we get a second view of the groups of the regular polyhedrons. Chapters 11 and 12 contain the author's own powerful method of handling the groups in three variables. As compared with the older methods of Jordan, Valentiner, and others, it is remarkable for its brevity and completeness. An important rôle is played by Kronecker's theorem on roots of unity, and also by the theorem that no prime > 7 can divide the order of a primitive group in three variables. In Chapter 13 we have a brief introduction to the theory of group characteristics and one or two of its applications.

The last formula on page 230 would be somewhat clearer if it were stated that α_i is a multiplier of T_i and that the notation $\alpha_i(y_1, y_2)$ means $(\alpha_i y_1, \alpha_i y_2)$. A misleading reference to "regular polygons" is made on page 219, line 14; for of course in the case of the dihedral groups the polygons are not regular. An unusual and rather unfortunate nomenclature is employed in § 77, line 1, and also in Part III, on page 373, line 7. While a set of values of the unknowns satisfying a system of equations, one for each unknown, is commonly called a solution of the system, the authors call it a set of solutions, the value of each unknown being a solution. In the definition on page 194 the condition that the equations "can be solved" for the new variables is hardly drastic enough; what is meant is that the equations must possess exactly one solution (or set of solutions, to use the author's language).

Part III, consisting of Chapters 14–20, begins with the Galois theory of equations, the treatment of which is somewhat similar to that of the author's excellent "Theory of Algebraic Equations," except that it is simpler and briefer, and that the comparatively easy case in which the domain of rationality involves only constants is wisely given first. Chapters 15 and 16, on the necessary and sufficient conditions for solvability by radicals, including a discussion of the cyclotomic equation and of the general cubic and quartic, naturally connect up with the chapter on solvable groups in Part I.

We now come to some interesting and valuable applications which show what a powerful tool the theory of groups is in solving many problems in geometry. In Chapter 17 we have constructions with ruler and compasses, in Chapter 18 the nine inflection points of a plane cubic curve, and in Chapter 19 the twenty-seven straight lines on a cubic surface as determined by means of their double-six configurations and the twenty-eight bitangents to a quartic curve as determined by means of their Steiner sets and Aronhold sets. The intimate connection between the last two problems is well shown. For the bitangents there is given an exhibition of Hesse's and Cayley's unsymmetric notation and also a purely algebraic derivation of the symmetric notation which was originally derived from the properties of theta functions. The methods of proof are largely the author's own and as compared with earlier treatments like that in Weber's Algebra, for instance, they show a vast improvement. Especially noteworthy is the determination of the number of real elements in each case.

The necessary geometric theorems are not presupposed, but are derived in the text; so that the discussion is intelligible to a reader with no knowledge of homogeneous coordinates or projective geometry; or at least it would be intelligible to him if some explanation had been given of the so-called points and lines at infinity, and if the cases in which these *ideal* elements are included had been carefully distinguished from those in which they are not. This has not been done; hence a certain amount of confusion. E. g., in § 176, fourth sentence, we are supposed to pass from a triangle of reference to Cartesian coordinates by setting $x_2/x_1 = x$, $x_3/x_1 = y$. But this special kind of triangle of reference has not been defined in the text. Moreover, in the definition of the ordinary triangle of reference, on page 327, the condition $\Delta \neq 0$ is not sufficient, for it would allow two sides of the triangle to be parallel. Again, on page 346, line 1, the surface may be a hyperbolic paraboloid as well as a hyperboloid of one sheet, even when the lines c_2, c_3, c_4 are all real. In the first paragraph of § 187, in order to avoid infinite values of m or ξ, the x-axis would have to be chosen so as to be neither parallel nor perpendicular to any bitangent.

The following typographical errors have been noticed.

Page 24, second footnote, second line. For "her" read "here."
Page 48, line 26. For "s_1" read "s_3."

Page 68, line 21. For "congredient" read "cogredient."
Page 140, line 9. Interchange "order" and "product."
Page 163, line 24. For "operator" read "operators."
Page 183, line 24. For "subgroups" read "subgroup."
§ 91, line 2. For the second "w" read "$\overline{w}$."
Page 212, exercise 1. After "transformation" insert "of determinant unity."
Page 345, footnote, line 5. For "c_3" read "c_4."
Page 381, line 6 from bottom. For "tenary" read "ternary."

In spite of a few slight blemishes, the book is an extremely valuable contribution to mathematical literature, one that for many years to come will be indispensable to the worker in the domain of finite groups.

ARTHUR RANUM.

SHORTER NOTICES.

L'Annuaire du Bureau des Longitudes pour l'An 1917. Paris, Gauthier-Villars.

AN interesting feature of the *Annuaire* for 1917 is the extensive "Notice"—"L'Avance de l'heure légale pendant l'été de l'année 1916," by M. J. Renaud. In its ninety-odd pages he traces the history of daylight-saving from the initiation of the mean time of Paris as the legal hour for France and Algeria to the passing of the law last year which put the clocks forward one hour during the summer months. A resumé of the opinions of scientific bodies, of the press, of the Chamber of Commerce of Paris, and even of the Académie des Sports on the last change is followed by reports of the discussions which took place in the Senate and Chamber of Deputies; there is much material here for those who are interested in the possible adoption of a similar change on this continent. A further section is devoted to the various measures which were taken to avoid confusion in making the change of hour in the spring and fall, and still another section to what has been done in other countries. M. Renaud concludes finally that concerted measures by international agreement are necessary if proper stability in the indication of legal time is to be effected.

In the other "Notices" M. G. Bigourdan gives an account of the Babylonian calendar in some detail, M. M. Hamy

sums up what has been done to determine the length of the meter in terms of wave lengths of light, and M. Renaud writes of the life and work of the hydrographic engineer, Ph. Hatt, who died in October, 1915.

The body of the volume contains a few changes which are mainly minor additions to and improvements in the astronomical portions which fill over half its pages in the odd years. No lessening of the care which has always been bestowed in making the *Annuaire* useful and up to date is apparent.

ERNEST W. BROWN.

Theory of Errors and Least Squares. By LEROY D. WELD. New York, Macmillan, 1916.

PROFESSOR WELD'S volume is mainly intended for those research workers who wish to use the method of least squares in dealing with their observations, but who have little knowledge of mathematics. With this end in view, he has attempted to give the necessary formulas in the simplest possible form and has added numerous examples drawn from problems in chemistry, physics, astronomy, geodesy, statistics, etc. Recognizing, however, that the competent observer is rarely willing to use a formula without knowing the basis on which it is constructed, the author devotes the earlier part to a detailed and simple explanation of the principles on which the theory is constructed.

He has attained considerable success in this effort. Nevertheless there is one point of view on which more explanation seems to be advisable. The theory of errors of observation, based on the least square principle, has become so standardized that those who use it much rarely need to remember that its results and technical terms rest on a set of assumptions which, it is true, are founded on experience, but which have not the definite character of a physical law deduced from observations. The beginner, whether an observer or a student learning the subject, needs to have the fact impressed on his mind that the whole theory with its technical terms such as "probable error of an observation," "most probable value" very largely form a language through which an observer may communicate briefly the results of his work and the degree of consistency of his observations. The theory gives no assurance that physical laws are correctly represented. This

is perhaps rather a point of view than a criticism, but it is one which frequently creates difficulties and its absence from most of the published treatments of the theory of errors is partly responsible for the unwillingness of many observers to use the methods of least squares.

A useful feature of the book is the Appendix, which contains some of the mathematical deductions which, placed in the body of the work, would perhaps have repelled or frightened the student. The various technical terms, rules and formulas are also gathered together ready for use.

ERNEST W. BROWN.

CARSLAW'S NON-EUCLIDEAN GEOMETRY.

I SHOULD like to point out that Professor Coolidge has quite misunderstood the definition of "nominal length" to which he refers on page 466 of his review of my little book on Non-Euclidean Geometry in the July BULLETIN; and he has failed to notice the indication I give at the beginning of § 94 of "just how a nominal line *corresponds* to a rectilinear segment."

As a matter of fact the full discussion of the euclidean case was given in my paper in the *Proceedings of the Edinburgh Mathematical Society*, and in Appendix V to the English translation of Bonola's book, both of which are mentioned in the footnote to page 156 of the book under review. It seemed unnecessary to repeat this introductory passage in full. In giving an abstract of it, the process of condensation has obviously been carried too far.

But a glance at one or other of the passages referred to will show that I am not guilty of the "lamentable" error with which your reviewer credits me.

H. S. CARSLAW.

THE UNIVERSITY OF SYDNEY,
August 10, 1917.

NOTES.

THE annual meeting of the American Mathematical Society will be held in New York City on Thursday and Friday, December 27–28. Titles and abstracts of papers intended for presentation at this meeting should be in the hands of the Secretary by December 3, if the abstracts are to be printed in advance of the meeting. Other titles may be sent in not later than December 10.

THE fortieth regular meeting of the Chicago Section will be held at the University of Chicago on Friday and Saturday, December 28–29. Friday afternoon will be devoted to a joint meeting with the Mathematical Association of America, at which Professor W. B. FORD will deliver his retiring address as chairman of the Section. The joint dinner of the two organizations will be held on Thursday evening. Titles and abstracts of papers intended for presentation at this meeting should be in the hands of the Secretary of the Section by December 3.

THE concluding (October) number of volume 18 of the *Transactions of the American Mathematical Society* contains the following papers: "Expansion problems of ordinary linear differential equations with auxiliary conditions at more than two points," by C. E. WILDER; "A memoir on the doctrine of associated forms," by O. E. GLENN; "The degree and class of multiply transitive groups," by W. A. MANNING; "Some geometric characterizations of isothermal nets on a curved surface," by G. M. GREEN; "A new method in boundary problems for differential equations," by R. G. D. RICHARDSON; "Note supplementary to the paper 'On certain pairs of transcendental functions whose roots separate each other,'" by MAXIME BÔCHER; "A theorem for space analogous to Cesàro's theorem for plane isogonal systems," by P. F. SMITH.

THE concluding (October) number of volume 39 of the *American Journal of Mathematics* contains the following papers: "The potential of a lens, and allied physical problems," by

G. GREENHILL; "On the asymptotic character of functions defined by series of the form $\Sigma c_n g(x+n)$," by R. D. CARMICHAEL; "Possible characteristic operators of a group," by G. A. MILLER; "Linear differential equations in infinitely many variables," by W. L. HART; "On the asymptotic value of sums of power residues," by H. H. MITCHELL; "On the structure of finite continuous groups," by J. A. BULLARD.

THE following mathematical papers have appeared in recent numbers of the *Proceedings of the National Academy of Sciences:* in the April, 1917, number (volume 3, number 4), "Dynamical systems with two degrees of freedom," by G. D. BIRKHOFF; in the July number (volume 3, number 7), "The Cayleyan curve of the quartic," by TERESA COHEN, and "Triads of transformations of conjugate systems of curves," by L. P. EISENHART; in the October number (volume 3, number 10), "On the general theory of curved surfaces and rectilinear congruences," by G. M. GREEN.

MANUSCRIPTS intended for publication in the *Bulletin de la Société mathématique de France* may hereafter be written in French, English, or Italian.

THE library of the American Mathematical Association has been deposited in the library of Oberlin College.

THE following 29 doctorates with mathematics as major subject were conferred by American universities in the academic year 1916–1917; the title of the dissertation is added in each case: C. C. BRAMBLE, Johns Hopkins, "A complete system for a collineation group isomorphic with the group of the double tangents of a plane quartic"; C. C. CAMP, Cornell, "An extension of the Sturm-Liouville expansion"; SARAH E. CRONIN, Columbia, "Geometric properties completely characterizing all the curves in a plane along which the constrained motion is such that the pressure is proportional to the normal component of the acting force"; W. L. CRUM, Yale, "The perturbations caused by a close approach of two asteroids"; MARY F. CURTIS, Harvard, "Curves invariant under point transformations of special type"; T. DANTZIG, Indiana, "Contributions to the theory of plane transformations"; L. R. FORD, Harvard, "On rational approximations to an irrational complex number"; E. D. GRANT, Chicago, "The motion of a flexible cable in a vertical plane"; C. M.

HEBBERT, Illinois, "Projective pencils of stelloids and the curve generated by them"; A. W. HOBBS, Johns Hopkins, "On a problem of projectiles"; T. R. HOLLCROFT, Cornell, "On the types of plane (2, 3) correspondences"; GOLDIE P. HORTON, Texas, "Lebesgue integrals"; M. T. HU, Harvard, "Linear integro-differential equations with a boundary condition"; E. P. HUBBLE, Chicago, "Photographic investigations of faint nebulæ"; K. W. LAMSON, Chicago, "A general implicit function theorem with an application to problems of relative minimum"; GILLIE A. LAREW, Chicago, "Necessary conditions for the problem of Mayer in the calculus of variations"; FLORA E. LESTOURGEON, Chicago, "Minima of functions of lines"; A. S. MERRILL, Chicago, "An isoperimetric problem with variable end points"; B. E. MITCHELL, Columbia, "Complex conics and their real representation"; H. C. M. MORSE, Harvard, "Certain types of geodesic motion on a surface of negative curvature"; G. W. MULLINS, Columbia, "Differential invariants under the inversion group"; I. C, NICHOLS, Michigan, "A comparative study on fractions in the early treatises on the Hindu art of reckoning"; J. F. RITT, Columbia, "On a general class of linear homogeneous differential equations of infinite order with constant coefficients preceded by a note on the resolution into partial fractions of the reciprocal of an entire function of genus zero"; E. F. SIMONDS, Columbia, "Differential invariants in the plane"; T. McN. SIMPSON, Chicago, "Relations between the metric and projective theories of space curves"; D. M. SMITH, Chicago, "Jacobi's condition for problems of Lagrange in the calculus of variations"; G. W. SMITH, Illinois, "On nilpotent algebras generated by two units such that i^2 is not an independent unit"; E. A. WHITE, Cornell, "Mechanical theory of the plow"; W. H. WILSON, Illinois, "On a certain class of functional equations."

RICE INSTITUTE. The following advanced mathematical courses are given during the present academic year.—By Professors P. J. DANIELL and G. C. EVANS and Dr. F. D. MURNAGHAN: Theory of functions of a real variable, three hours; Aerodynamics, three hours.—By Professor DANIELL: Statistics and probability, three hours.—By Professor EVANS: Differential and integral equations, three hours.—By Professor W. C. GRAUSTEIN: Projective geometry, three hours.

THE following is an additional list of university and college teachers of mathematics who have recently entered the national military service:

Major D. H. CRISSY, professor of mathematics at West Point, has been appointed commandant of the government school of aeronautics at Princeton. Professor C. W. COBB, of Amherst College, has been granted a year's leave of absence to enter the aviation work of the government. Professor W. J. BERRY, of the Brooklyn Polytechnic Institute, has been appointed second lieutenant 308th infantry, U. S. N. A. Professor E. I. SHEPARD, of Williams College, has been commissioned captain in the officers' reserve corps. Professor J. A. NYSWANDER, of the University of Nevada, Professor C. C. CAMP, of Ottawa University, Dr. L. M. KELLS, of the University of Illinois, and Dr. J. M. STETSON, of Adelbert College, have joined the national army. Dr. L. L. SILVERMAN, of Cornell University, has accepted a position with the Massachusetts board of public safety.

AT Purdue University associate professor WILLIAM MARSHALL has been promoted to a full professorship of mathematics, and Mr. GLENN JAMES has been promoted to an assistant professorship of mathematics. Professors G. ALFORD and J. WESTLUND have retired from active service.

AT the University of Michigan junior professor W. B. FORD has been promoted to a full professorship of mathematics, and Drs. L. A. HOPKINS and V. C. POOR have been promoted to assistant professorships of mathematics. Mr. G. H. CRESSE, Mr. G. D. JONES, and Dr. R. B. ROBBINS have been appointed instructors in mathematics. Dr. V. H. WELLS has been appointed professor of mathematics and statistics in the University of Pittsburgh, and Dr. A. L. MILLER has been appointed instructor in mathematics in Harvard University.

PROFESSOR C. N. HASKINS, of Dartmouth College, has been granted leave of absence for the present half-year.

AT the University of Pennsylvania Professor O. E. GLENN is absent on leave during the present half-year. Dr. J. R. KLINE has been appointed instructor in mathematics.

DR. HARRY BATEMAN has been appointed professor of aeronautical research and mathematical physics in Throop College.

AT the University of Texas adjunct professor J. W. CALHOUN has been promoted to an associate professorship of pure mathematics, and adjunct professor E. L. DODD has been promoted to an associate professorship of actuarial mathematics. Mr. H. J. ETTLINGER has been promoted to an adjunct professorship of applied mathematics. Dr. T. M. SIMPSON has been appointed to an adjunct professorship of mathematics.

MR. E. H. CLARKE, of Purdue University, has been appointed professor of mathematics in Hiram College.

DR. D. D. LEIB, of Connecticut College, has been promoted to an associate professorship of mathematics.

AT Iowa State College Dr. A. HELEN TAPPAN has been promoted to an assistant professorship of mathematics.

AT the Carnegie Institute of Technology at Pittsburgh Dr. J. C. MOREHEAD has been promoted to an assistant professorship of mathematics and descriptive geometry.

MR. I. L. MILLER, of Indiana University, has been appointed professor of mathematics in Carthage College.

DR. E. F. SIMONDS, of Columbia University, has been appointed instructor in mathematics in the University of Illinois.

DR. PAULINE SPERRY, of Smith College, has been appointed instructor in mathematics in the University of California.

MR. H. P. KEAN has been appointed instructor in mathematics in Milliken University.

DR. W. H. WILSON has been appointed instructor in mathematics in the Massachusetts Institute of Technology.

DR. G. W. SMITH has been appointed instructor in mathematics and acting head of the department of physics in Beloit College.

DR. C. C. BRAMBLE has been appointed instructor in mathematics in the U. S. Naval Academy.

DR. A. W. HOBBS has been appointed instructor in mathematics in the University of North Carolina.

DR. C. M. HEBBERT and Dr. F. W. REED have been appointed instructors in mathematics in the aviation school of the University of Illinois.

DR. R. W. BURGESS, of Brown University, is now in the Bureau of Statistics at Washington.

DR. RUTH GENTRY died at Indianapolis, Ind., October 18, 1917. Miss Gentry had been a member of the American Mathematical Society since 1894.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

BOUTROUX (P.). See PASCAL (B.).

BRUNSCHVICG (L.). See PASCAL (B.).

BURKHARDT (H.). See ENCYCLOPÉDIE.

ENCYCLOPÉDIE des sciences mathématiques pures et appliquées. Tome II, volume 4, fascicule 2: Groupes de transformations continus; exposé, d'après l'article allemand de H. Burkhardt et L. Maurer, par E. Vessiot. Paris, Gauthier-Villars, 1916.

GAZIER (F.). See PASCAL (B.).

GEORGI (K.). Ueber die konforme Abbildung gewisser nichtsymmetrischer unendlichvielfach zusammenhängender Bereiche auf Kreisbereiche. (Diss.) Jena, 1915.

GMEINER (J. A.). See STOLZ (O.).

GROTENDORST (N. C.). Beginselen der differentiaal- en integraalrekening. Vierde verbeterde druk. Breda, De Koninklijke Militaire Academie, 1916. Gr. 8vo. 10 + 551 pp.

HARTMANN (F.). Zur Lösung des Fermatschen Problems. Leipzig, 1916. M. 1.00

KORTMULDER (R. J.). De logische grondslagen der wiskunde. (Diss.) Amsterdam, A. H. Kruyt, 1916.

MAURER (L.). See ENCYCLOPÉDIE.

MERCHICH (M.). Organistik der Geometrie. Grundzüge der geometrischen Prinzipienlehre, im Gegensatz zur euklidischen und nichteuklidischen Kasuistik. Moson, Ungarn, Selbstverlag, 1914. Gr. 8vo. 256 pp.

NABER (H. A.). Meetkunde en mystiek. Drie voordrachten. Amsterdam, N. V. Theosofische Uitgevers-Maatschappij, 1915. Gr. 4to. 99 pp.+7 plates. Fl. 4.50

PASCAL (B.). Oeuvres. Publiées suivant l'ordre chronologique avec documents complémentaires, introductions et notes par L. Brunschvicg, P. Boutroux, et F. Gazier. 14 tomes. Paris, Hachette, 1908–1914. 8vo. Chaque volume, Fr. 7.50

SCHUH (F.). Grepen uit de moderne meetkunde. (Noordhoff's Verzameling van Wiskundige Werken, Deel 2.) Eerste deel: Reciproke transformaties in het vlak en in de ruimte. Hyperboloïden en kegelsneden. Harmonische eigenschappen en cirkelbundels. Groningen, Noordhoff, 1916. Gr. 8vo. 24+590 pp.

STOLZ (O.) und GMEINER (J. A.). Theoretische Arithmetik. 2te Abteilung: Die Lehren von den reellen und von den komplexen Zahlen. 2te Auflage. Leipzig, Teubner, 1915. 8+369 pp. M. 12.00

VESSIOT (E.). See ENCYCLOPÉDIE.

II. ELEMENTARY MATHEMATICS.

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